One Variable Advanced Calculus

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Contents

Introduction

The difference between advanced calculus and calculus is that all the theorems are proved completely and the role of plane geometry is minimized. Instead, the notion of completeness is of preeminent importance. Silly gimmicks are of no significance at all. Routine skills involving elementary functions and integration techniques are supposed to be mastered and have no place in advanced calculus which deals with the fundamental issues related to existence and meaning. This is a subject which places calculus as part of mathematics and involves proofs and definitions, not algorithms and busy work.

An orderly development of the elementary functions is included but it is assumed the reader is familiar enough with these functions to use them in problems which illustrate some of the ideas presented.

The Real And Complex Numbers

2.1 The Number Line And Algebra Of The Real Numbers

To begin with, consider the real numbers, denoted by $\mathbb{R}$, as a line extending infinitely far in both directions. In this book, the notation, $\equiv$ indicates something is being defined. Thus the integers are defined as

$$\mathbb{Z} \equiv \{\cdots - 1, 0, 1, \cdots\},$$

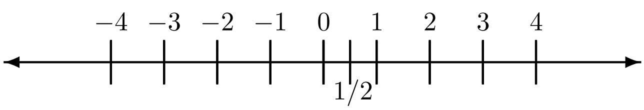
the natural numbers,

$$\mathbb{N} \equiv \{1, 2, \cdots\}$$

and the rational numbers, defined as the numbers which are the quotient of two integers.

$$\mathbb{Q} \equiv \left\{ \frac{m}{n} \text{ such that } m, n \in \mathbb{Z}, n \neq 0 \right\}$$

are each subsets of $\mathbb{R}$ as indicated in the following picture.



As shown in the picture, $\frac{1}{2}$ is half way between the number 0 and the number, 1. By analogy, you can see where to place all the other rational numbers. It is assumed that $\mathbb{R}$ has the following algebra properties, listed here as a collection of assertions called axioms. These properties will not be proved which is why they are called axioms rather than theorems. In general, axioms are statements which are regarded as true. Often these are things which are "self evident" either from experience or from some sort of intuition but this does not have to be the case.

Axiom 2.1.1 $x + y = y + x$, *(commutative law for addition)*

Axiom 2.1.2 $x + 0 = x$, *(additive identity).*

Axiom 2.1.3 *For each* $x \in \mathbb{R}$, *there exists* $-x \in \mathbb{R}$ *such that* $x + (-x) = 0$, *(existence of additive inverse).*

Axiom 2.1.4 $(x+y)+z=x+(y+z)$, (*associative law for addition*).

Axiom 2.1.5 $xy=yx$, (*commutative law for multiplication*).

Axiom 2.1.6 $(xy)z=x(yz)$, (*associative law for multiplication*).

Axiom 2.1.7 $1x=x$, (*multiplicative identity*).

Axiom 2.1.8 *For each* $x\neq 0$, *there exists* x^{-1} *such that* $xx^{-1}=1$.(*existence of multiplicative inverse*).

Axiom 2.1.9 $x(y+z)=xy+xz$.(*distributive law*).

These axioms are known as the field axioms and any set (there are many others besides $\mathbb{R}$) which has two such operations satisfying the above axioms is called a field. Division and subtraction are defined in the usual way by $x-y\equiv x+(-y)$ and $x/y\equiv x\left(y^{-1}\right)$. It is assumed that the reader is completely familiar with these axioms in the sense that he or she can do the usual algebraic manipulations taught in high school and junior high algebra courses. The axioms listed above are just a careful statement of exactly what is necessary to make the usual algebraic manipulations valid. A word of advice regarding division and subtraction is in order here. Whenever you feel a little confused about an algebraic expression which involves division or subtraction, think of division as multiplication by the multiplicative inverse as just indicated and think of subtraction as addition of the additive inverse. Thus, when you see x/y, think $x\left(y^{-1}\right)$ and when you see $x-y$, think $x+(-y)$. In many cases the source of confusion will disappear almost magically. The reason for this is that subtraction and division do not satisfy the associative law. This means there is a natural ambiguity in an expression like $6-3-4$. Do you mean $(6-3)-4=-1$ or $6-(3-4)=6-(-1)=7$? It makes a difference doesn't it? However, the so called binary operations of addition and multiplication are associative and so no such confusion will occur. It is conventional to simply do the operations in order of appearance reading from left to right. Thus, if you see $6-3-4$, you would normally interpret it as the first of the above alternatives.

In doing algebra, the following theorem is important and follows from the above axioms. The reasoning which demonstrates this assertion is called a proof. Proofs and definitions are very important in mathematics because they are the means by which "truth" is determined. In mathematics, something is "true" if it follows from axioms using a correct logical argument. Truth is not determined on the basis of experiment or opinions and it is this which makes mathematics useful as a language for describing certain kinds of reality in a precise manner.[1] It is also the definitions and proofs which make the subject of mathematics intellectually worth while. Take these away and it becomes a gray wasteland filled with endless tedium and meaningless manipulations.

In the first part of the following theorem, the claim is made that the additive inverse and the multiplicative inverse are unique. This means that for a given number, only one number has the property that it is an additive inverse and that, given a nonzero number, only one number has the property that it is a multiplicative inverse. The significance of this is that if you are wondering if a given number is the additive inverse of a given number, all you have to do is to check and see if it acts like one.

Theorem 2.1.10 *The above axioms imply the following.*

1. *The multiplicative inverse and additive inverses are unique.*

[1] There are certainly real and important things which should not be described using mathematics because it has nothing to do with these things. For example, feelings and emotions have nothing to do with math.

2. $0x = 0, \ -(-x) = x,$

3. $(-1)(-1) = 1, \ (-1)x = -x$

4. If $xy = 0$ *then either* $x = 0$ *or* $y = 0$.

Proof: Suppose then that x is a real number and that $x + y = 0 = x + z$. It is necessary to verify $y = z$. From the above axioms, there exists an additive inverse, $-x$ for x. Therefore,

$$-x + 0 = (-x) + (x + y) = (-x) + (x + z)$$

and so by the associative law for addition,

$$((-x) + x) + y = ((-x) + x) + z$$

which implies

$$0 + y = 0 + z.$$

Now by the definition of the additive identity, this implies $y = z$. You should prove the multiplicative inverse is unique.

Consider 2. It is desired to verify $0x = 0$. From the definition of the additive identity and the distributive law it follows that

$$0x = (0 + 0)x = 0x + 0x.$$

From the existence of the additive inverse and the associative law it follows

$$\begin{aligned} 0 &= (-0x) + 0x = (-0x) + (0x + 0x) \\ &= ((-0x) + 0x) + 0x = 0 + 0x = 0x \end{aligned}$$

To verify the second claim in 2., it suffices to show x acts like the additive inverse of $-x$ in order to conclude that $-(-x) = x$. This is because it has just been shown that additive inverses are unique. By the definition of additive inverse,

$$x + (-x) = 0$$

and so $x = -(-x)$ as claimed.

To demonstrate 3.,

$$(-1)(1 + (-1)) = (-1)0 = 0$$

and so using the definition of the multiplicative identity, and the distributive law,

$$(-1) + (-1)(-1) = 0.$$

It follows from 1. and 2. that $1 = -(-1) = (-1)(-1)$. To verify $(-1)x = -x$, use 2. and the distributive law to write

$$x + (-1)x = x(1 + (-1)) = x0 = 0.$$

Therefore, by the uniqueness of the additive inverse proved in 1., it follows $(-1)x = -x$ as claimed.

To verify 4., suppose $x \neq 0$. Then x^{-1} exists by the axiom about the existence of multiplicative inverses. Therefore, by 2. and the associative law for multiplication,

$$y = \left(x^{-1}x\right)y = x^{-1}(xy) = x^{-1}0 = 0.$$

This proves 4. and completes the proof of this theorem.

Recall the notion of something raised to an integer power. Thus $y^2 = y \times y$ and $b^{-3} = \frac{1}{b^3}$ etc.

Also, there are a few conventions related to the order in which operations are performed. Exponents are always done before multiplication. Thus $xy^2 = x\left(y^2\right)$ and is not equal to $(xy)^2$. Division or multiplication is always done before addition or subtraction. Thus $x - y(z+w) = x - [y(z+w)]$ and is not equal to $(x-y)(z+w)$. Parentheses are done before anything else. Be very careful of such things since they are a source of mistakes. When you have doubts, insert parentheses to resolve the ambiguities.

Also recall summation notation. If you have not seen this, the following is a short review of this topic.

Definition 2.1.11 *Let $x_1, x_2, \cdots, x_m$ be numbers. Then*

$$\sum_{j=1}^{m} x_j \equiv x_1 + x_2 + \cdots + x_m.$$

Thus this symbol, $\sum_{j=1}^{m} x_j$ means to take all the numbers, $x_1, x_2, \cdots, x_m$ and add them all up. Note the use of the j as a generic variable which takes values from 1 up to m. This notation will be used whenever there are things which can be added, not just numbers.

As an example of the use of this notation, you should verify the following.

Example 2.1.12 $\sum_{k=1}^{6} (2k+1) = 48.$

Be sure you understand why

$$\sum_{k=1}^{m+1} x_k = \sum_{k=1}^{m} x_k + x_{m+1}.$$

As a slight generalization of this notation,

$$\sum_{j=k}^{m} x_j \equiv x_k + \cdots + x_m.$$

It is also possible to change the variable of summation.

$$\sum_{j=1}^{m} x_j = x_1 + x_2 + \cdots + x_m$$

while if r is an integer, the notation requires

$$\sum_{j=1+r}^{m+r} x_{j-r} = x_1 + x_2 + \cdots + x_m$$

and so $\sum_{j=1}^{m} x_j = \sum_{j=1+r}^{m+r} x_{j-r}$.

Summation notation will be used throughout the book whenever it is convenient to do so.

Another thing to keep in mind is that you often use letters to represent numbers. Since they represent numbers, you manipulate expressions involving letters in the same manner as you would if they were specific numbers.

Example 2.1.13 *Add the fractions,* $\frac{x}{x^2+y}+\frac{y}{x-1}$.

You add these just like they were numbers. Write the first expression as $\frac{x(x-1)}{(x^2+y)(x-1)}$ and the second as $\frac{y(x^2+y)}{(x-1)(x^2+y)}$. Then since these have the same common denominator, you add them as follows.

$$\begin{aligned}\frac{x}{x^2+y}+\frac{y}{x-1} &= \frac{x(x-1)}{(x^2+y)(x-1)}+\frac{y\left(x^2+y\right)}{(x-1)(x^2+y)} \\ &= \frac{x^2-x+yx^2+y^2}{(x^2+y)(x-1)}.\end{aligned}$$

2.2 Exercises

1. Consider the expression $x+y(x+y)-x(y-x)\equiv f(x,y)$. Find $f(-1,2)$.
2. Show $-(ab)=(-a)b$.
3. Show on the number line the effect of adding two positive numbers, x and y.
4. Show on the number line the effect of subtracting a positive number from another positive number.
5. Show on the number line the effect of multiplying a number by -1.
6. Add the fractions $\frac{x}{x^2-1}+\frac{x-1}{x+1}$.
7. Find a formula for $(x+y)^2, (x+y)^3$, and $(x+y)^4$. Based on what you observe for these, give a formula for $(x+y)^8$.
8. When is it true that $(x+y)^n=x^n+y^n$?
9. Find the error in the following argument. Let $x=y=1$. Then $xy=y^2$ and so $xy-x^2=y^2-x^2$. Therefore, $x(y-x)=(y-x)(y+x)$. Dividing both sides by $(y-x)$ yields $x=x+y$. Now substituting in what these variables equal yields $1=1+1$.
10. Find the error in the following argument. $\sqrt{x^2+1}=x+1$ and so letting $x=2$, $\sqrt{5}=3$. Therefore, $5=9$.
11. Find the error in the following. Let $x=1$ and $y=2$. Then $\frac{1}{3}=\frac{1}{x+y}=\frac{1}{x}+\frac{1}{y}=1+\frac{1}{2}=\frac{3}{2}$. Then cross multiplying, yields $2=9$.
12. Simplify $\frac{x^2y^4z^{-6}}{x^{-2}y^{-1}z}$.
13. Simplify the following expressions using correct algebra. In these expressions the variables represent real numbers.
    (a) $\frac{x^2y+xy^2+x}{x}$
    (b) $\frac{x^2y+xy^2+x}{xy}$
    (c) $\frac{x^3+2x^2-x-2}{x+1}$
14. Find the error in the following argument. Let $x=3$ and $y=1$. Then $1=3-2=3-(3-1)=x-y(x-y)=(x-y)(x-y)=2^2=4$.

15. Verify the following formulas.

    (a) $(x-y)(x+y) = x^2 - y^2$
    (b) $(x-y)\left(x^2+xy+y^2\right) = x^3 - y^3$
    (c) $(x+y)\left(x^2-xy+y^2\right) = x^3 + y^3$

16. Find the error in the following.

$$\frac{xy+y}{x} = y + y = 2y.$$

    Now let $x = 2$ and $y = 2$ to obtain

$$3 = 4$$

17. Show the rational numbers satisfy the field axioms. You may assume the associative, commutative, and distributive laws hold for the integers.

2.3 Set Notation

A set is just a collection of things called elements. Often these are also referred to as points in calculus. For example $\{1,2,3,8\}$ would be a set consisting of the elements 1,2,3, and 8. To indicate that 3 is an element of $\{1,2,3,8\}$, it is customary to write $3 \in \{1,2,3,8\}$. $9 \notin \{1,2,3,8\}$ means 9 is not an element of $\{1,2,3,8\}$. Sometimes a rule specifies a set. For example you could specify a set as all integers larger than 2. This would be written as $S = \{x \in \mathbb{Z} : x > 2\}$. This notation says: the set of all integers, x, such that $x > 2$.

If A and B are sets with the property that every element of A is an element of B, then A is a subset of B. For example, $\{1,2,3,8\}$ is a subset of $\{1,2,3,4,5,8\}$, in symbols, $\{1,2,3,8\} \subseteq \{1,2,3,4,5,8\}$. The same statement about the two sets may also be written as $\{1,2,3,4,5,8\} \supseteq \{1,2,3,8\}$.

The union of two sets is the set consisting of everything which is contained in at least one of the sets, A or B. As an example of the union of two sets, $\{1,2,3,8\} \cup \{3,4,7,8\} = \{1,2,3,4,7,8\}$ because these numbers are those which are in at least one of the two sets. In general

$$A \cup B \equiv \{x : x \in A \text{ or } x \in B\}.$$

Be sure you understand that something which is in both A and B is in the union. It is not an exclusive or.

The intersection of two sets, A and B consists of everything which is in both of the sets. Thus $\{1,2,3,8\} \cap \{3,4,7,8\} = \{3,8\}$ because 3 and 8 are those elements the two sets have in common. In general,

$$A \cap B \equiv \{x : x \in A \text{ and } x \in B\}.$$

When with real numbers, $[a,b]$ denotes the set of real numbers, x, such that $a \leq x \leq b$ and $[a,b)$ denotes the set of real numbers such that $a \leq x < b$. (a,b) consists of the set of real numbers, x such that $a < x < b$ and $(a,b]$ indicates the set of numbers, x such that $a < x \leq b$. $[a,\infty)$ means the set of all numbers, x such that $x \geq a$ and $(-\infty,a]$ means the set of all real numbers which are less than or equal to a. These sorts of sets of real numbers are called intervals. The two points, a and b are called endpoints of the interval. Other intervals such as $(-\infty,b)$ are defined by analogy to what was just explained. In general, the curved parenthesis indicates the end point it sits next to is not included while the square parenthesis indicates this end point is included. The reason that there will always be a curved parenthesis next to ∞ or $-\infty$ is that these are not real numbers. Therefore, they cannot be included in any set of real numbers.

A special set which needs to be given a name is the empty set also called the null set, denoted by $\emptyset$. Thus $\emptyset$ is defined as the set which has no elements in it. Mathematicians like to say the empty set is a subset of every set. The reason they say this is that if it were not so, there would have to exist a set, A, such that $\emptyset$ has something in it which is not in A. However, $\emptyset$ has nothing in it and so the least intellectual discomfort is achieved by saying $\emptyset \subseteq A$.

If A and B are two sets, $A \setminus B$ denotes the set of things which are in A but not in B. Thus

$$A \setminus B \equiv \{x \in A : x \notin B\}.$$

Set notation is used whenever convenient.

2.4 Order

The real numbers also have an order defined on them. This order may be defined by reference to the positive real numbers, those to the right of 0 on the number line, denoted by $\mathbb{R}^+$ which is assumed to satisfy the following axioms.

$$\text{The sum of two positive real numbers is positive.} \tag{2.1}$$

$$\text{The product of two positive real numbers is positive.} \tag{2.2}$$

For a given real number, x, one and only one of the following alternatives holds. Either x is positive, x equals 0 or $-x$ is positive. (2.3)

Definition 2.4.1 *$x < y$ exactly when $y + (-x) \equiv y - x \in \mathbb{R}^+$. In the usual way, $x < y$ is the same as $y > x$ and $x \leq y$ means either $x < y$ or $x = y$. The symbol $\geq$ is defined similarly.*

Theorem 2.4.2 *The following hold for the order defined as above.*

1. *If $x < y$ and $y < z$ then $x < z$ (Transitive law).*
2. *If $x < y$ then $x + z < y + z$ (addition to an inequality).*
3. *If $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.*
4. *If $x > 0$ then $x^{-1} > 0$.*
5. *If $x < 0$ then $x^{-1} < 0$.*
6. *If $x < y$ then $xz < yz$ if $z > 0$, (multiplication of an inequality).*
7. *If $x < y$ and $z < 0$, then $xz > zy$ (multiplication of an inequality).*
8. *Each of the above holds with $>$ and $<$ replaced by $\geq$ and $\leq$ respectively except for 4 and 5 in which we must also stipulate that $x \neq 0$.*
9. *For any x and y, exactly one of the following must hold. Either $x = y$, $x < y$, or $x > y$ (trichotomy).*

Proof: First consider 1, the transitive law. Suppose $x < y$ and $y < z$. Why is $x < z$? In other words, why is $z - x \in \mathbb{R}^+$? It is because $z - x = (z - y) + (y - x)$ and both $z - y, y - x \in \mathbb{R}^+$. Thus by 2.1 above, $z - x \in \mathbb{R}^+$ and so $z > x$.

Next consider 2, addition to an inequality. If $x < y$ why is $x + z < y + z$? it is because

$$\begin{aligned}(y+z)+-(x+z) &= (y+z)+(-1)(x+z) \\ &= y+(-1)x+z+(-1)z \\ &= y-x\in\mathbb{R}^+.\end{aligned}$$

Next consider 3. If $x \leq 0$ and $y \leq 0$, why is $xy \geq 0$? First note there is nothing to show if either x or y equal 0 so assume this is not the case. By 2.3 $-x > 0$ and $-y > 0$. Therefore, by 2.2 and what was proved about $-x = (-1)x$,

$$(-x)(-y) = (-1)^2 xy \in \mathbb{R}^+.$$

Is $(-1)^2 = 1$? If so the claim is proved. But $-(-1) = (-1)^2$ and $-(-1) = 1$ because

$$-1 + 1 = 0.$$

Next consider 4. If $x > 0$ why is $x^{-1} > 0$? By 2.3 either $x^{-1} = 0$ or $-x^{-1} \in \mathbb{R}^+$. It can't happen that $x^{-1} = 0$ because then you would have to have $1 = 0x$ and as was shown earlier, $0x = 0$. Therefore, consider the possibility that $-x^{-1} \in \mathbb{R}^+$. This can't work either because then you would have

$$(-1)x^{-1}x = (-1)(1) = -1$$

and it would follow from 2.2 that $-1 \in \mathbb{R}^+$. But this is impossible because if $x \in \mathbb{R}^+$, then $(-1)x = -x \in \mathbb{R}^+$ and contradicts 2.3 which states that either $-x$ or x is in $\mathbb{R}^+$ but not both.

Next consider 5. If $x < 0$, why is $x^{-1} < 0$? As before, $x^{-1} \neq 0$. If $x^{-1} > 0$, then as before,

$$-x\left(x^{-1}\right) = -1 \in \mathbb{R}^+$$

which was just shown not to occur.

Next consider 6. If $x < y$ why is $xz < yz$ if $z > 0$? This follows because

$$yz - xz = z(y - x) \in \mathbb{R}^+$$

since both z and $y - x \in \mathbb{R}^+$.

Next consider 7. If $x < y$ and $z < 0$, why is $xz > zy$? This follows because

$$zx - zy = z(x - y) \in \mathbb{R}^+$$

by what was proved in 3.

The last two claims are obvious and left for you. This proves the theorem.

Note that trichotomy could be stated by saying $x \leq y$ or $y \leq x$.

Definition 2.4.3 $|x| \equiv \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$

Note that $|x|$ can be thought of as the distance between x and 0.

Theorem 2.4.4 $|xy| = |x|\,|y|$.

Proof: You can verify this by checking all available cases. Do so.

Theorem 2.4.5 *The following inequalities hold.*

$$|x+y| \leq |x| + |y|\,, \quad ||x| - |y|| \leq |x-y|\,.$$

Either of these inequalities may be called the triangle inequality.

Proof: First note that if $a, b \in \mathbb{R}^+ \cup \{0\}$ then $a \leq b$ if and only if $a^2 \leq b^2$. Here is why. Suppose $a \leq b$. Then by the properties of order proved above,

$$a^2 \leq ab \leq b^2$$

because $b^2 - ab = b(b-a) \in \mathbb{R}^+ \cup \{0\}$. Next suppose $a^2 \leq b^2$. If both $a, b = 0$ there is nothing to show. Assume then they are not both 0. Then

$$b^2 - a^2 = (b+a)(b-a) \in \mathbb{R}^+.$$

By the above theorem on order, $(a+b)^{-1} \in \mathbb{R}^+$ and so using the associative law,

$$(a+b)^{-1}((b+a)(b-a)) = (b-a) \in \mathbb{R}^+$$

Now

$$\begin{aligned} |x+y|^2 &= (x+y)^2 = x^2 + 2xy + y^2 \\ &\leq |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2 \end{aligned}$$

and so the first of the inequalities follows. Note I used $xy \leq |xy| = |x||y|$ which follows from the definition.

To verify the other form of the triangle inequality,

$$x = x - y + y$$

so

$$|x| \leq |x-y| + |y|$$

and so

$$|x| - |y| \leq |x-y| = |y-x|$$

Now repeat the argument replacing the roles of x and y to conclude

$$|y| - |x| \leq |y-x|\,.$$

Therefore,

$$||y| - |x|| \leq |y-x|\,.$$

This proves the triangle inequality.

Example 2.4.6 *Solve the inequality* $2x + 4 \leq x - 8$

Subtract $2x$ from both sides to yield $4 \leq -x-8$. Next add 8 to both sides to get $12 \leq -x$. Then multiply both sides by (-1) to obtain $x \leq -12$. Alternatively, subtract x from both sides to get $x + 4 \leq -8$. Then subtract 4 from both sides to obtain $x \leq -12$.

Example 2.4.7 *Solve the inequality* $(x+1)(2x-3) \geq 0$.

If this is to hold, either both of the factors, $x+1$ and $2x-3$ are nonnegative or they are both non-positive. The first case yields $x+1 \geq 0$ and $2x-3 \geq 0$ so $x \geq -1$ and $x \geq \frac{3}{2}$ yielding $x \geq \frac{3}{2}$. The second case yields $x+1 \leq 0$ and $2x-3 \leq 0$ which implies $x \leq -1$ and $x \leq \frac{3}{2}$. Therefore, the solution to this inequality is $x \leq -1$ or $x \geq \frac{3}{2}$.

Example 2.4.8 *Solve the inequality* $(x)(x+2) \geq -4$

Here the problem is to find x such that $x^2+2x+4 \geq 0$. However, $x^2+2x+4 = (x+1)^2+3 \geq 0$ for all x. Therefore, the solution to this problem is all $x \in \mathbb{R}$.

Example 2.4.9 *Solve the inequality* $2x+4 \leq x-8$

This is written as $(-\infty, -12]$.

Example 2.4.10 *Solve the inequality* $(x+1)(2x-3) \geq 0$.

This was worked earlier and $x \leq -1$ or $x \geq \frac{3}{2}$ was the answer. In terms of set notation this is denoted by $(-\infty, -1] \cup [\frac{3}{2}, \infty)$.

Example 2.4.11 *Solve the equation* $|x-1| = 2$

This will be true when $x-1=2$ or when $x-1=-2$. Therefore, there are two solutions to this problem, $x=3$ or $x=-1$.

Example 2.4.12 *Solve the inequality* $|2x-1| < 2$

From the number line, it is necessary to have $2x-1$ between -2 and 2 because the inequality says that the distance from $2x-1$ to 0 is less than 2. Therefore, $-2 < 2x-1 < 2$ and so $-1/2 < x < 3/2$. In other words, $-1/2 < x$ and $x < 3/2$.

Example 2.4.13 *Solve the inequality* $|2x-1| > 2$.

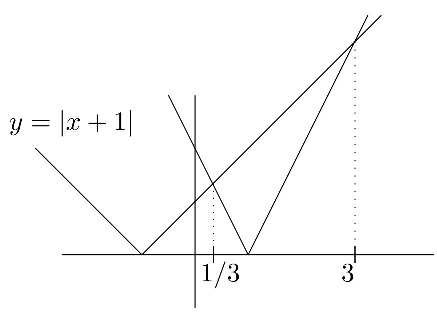
This happens if $2x-1 > 2$ or if $2x-1 < -2$. Thus the solution is $x > 3/2$ or $x < -1/2$. Written in terms of intervals this is $\left(\frac{3}{2}, \infty\right) \cup \left(-\infty, -\frac{1}{2}\right)$.

Example 2.4.14 *Solve* $|x+1| = |2x-2|$

There are two ways this can happen. It could be the case that $x+1 = 2x-2$ in which case $x=3$ or alternatively, $x+1 = 2-2x$ in which case $x = 1/3$.

Example 2.4.15 *Solve* $|x+1| \leq |2x-2|$

In order to keep track of what is happening, it is a very good idea to graph the two relations, $y = |x+1|$ and $y = |2x-2|$ on the same set of coordinate axes. This is not a hard job. $|x+1| = x+1$ when $x > -1$ and $|x+1| = -1-x$ when $x \leq -1$. Therefore, it is not hard to draw its graph. Similar considerations apply to the other relation. The result is



Equality holds exactly when $x = 3$ or $x = \frac{1}{3}$ as in the preceding example. Consider x between $\frac{1}{3}$ and 3. You can see these values of x do not solve the inequality. For example $x = 1$ does not work. Therefore, $\left(\frac{1}{3}, 3\right)$ must be excluded. The values of x larger than 3 do not produce equality so either $|x+1| < |2x-2|$ for these points or $|2x-2| < |x+1|$ for these points. Checking examples, you see the first of the two cases is the one which holds. Therefore, $[3, \infty)$ is included. Similar reasoning obtains $(-\infty, \frac{1}{3}]$. It follows the solution set to this inequality is $(-\infty, \frac{1}{3}] \cup [3, \infty)$.

Example 2.4.16 *Suppose $\varepsilon > 0$ is a given positive number. Obtain a number, $\delta > 0$, such that if $|x-1| < \delta$, then $\left|x^2 - 1\right| < \varepsilon$.*

First of all, note $\left|x^2-1\right| = |x-1|\,|x+1| \leq (|x|+1)\,|x-1|$. Now if $|x-1| < 1$, it follows $|x| < 2$ and so for $|x-1| < 1$,

$$\left|x^2 - 1\right| < 3\,|x-1|.$$

Now let $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$. This notation means to take the minimum of the two numbers, 1 and $\frac{\varepsilon}{3}$. Then if $|x-1| < \delta$,

$$\left|x^2 - 1\right| < 3\,|x-1| < 3\frac{\varepsilon}{3} = \varepsilon.$$

2.5 Exercises

1. Solve $(3x+2)(x-3) \leq 0$.
2. Solve $(3x+2)(x-3) > 0$.
3. Solve $\frac{x+2}{3x-2} < 0$.
4. Solve$\frac{x+1}{x+3} < 1$.
5. Solve $(x-1)(2x+1) \leq 2$.
6. Solve $(x-1)(2x+1) > 2$.
7. Solve $x^2 - 2x \leq 0$.
8. Solve $(x+2)(x-2)^2 \leq 0$.
9. Solve $\frac{3x-4}{x^2+2x+2} \geq 0$.
10. Solve $\frac{3x+9}{x^2+2x+1} \geq 1$.
11. Solve $\frac{x^2+2x+1}{3x+7} < 1$.
12. Solve $|x+1| = |2x-3|$.
13. Solve $|3x+1| < 8$. Give your answer in terms of intervals on the real line.
14. Sketch on the number line the solution to the inequality $|x-3| > 2$.
15. Sketch on the number line the solution to the inequality $|x-3| < 2$.
16. Show $|x| = \sqrt{x^2}$.
17. Solve $|x+2| < |3x-3|$.

18. Tell when equality holds in the triangle inequality.

19. Solve $|x+2| \leq 8 + |2x-4|$.

20. Solve $(x+1)(2x-2)x \geq 0$.

21. Solve $\frac{x+3}{2x+1} > 1$.

22. Solve $\frac{x+2}{3x+1} > 2$.

23. Describe the set of numbers, a such that there is no solution to $|x+1| = 4 - |x+a|$.

24. Suppose $0 < a < b$. Show $a^{-1} > b^{-1}$.

25. Show that if $|x-6| < 1$, then $|x| < 7$.

26. Suppose $|x-8| < 2$. How large can $|x-5|$ be?

27. Obtain a number, $\delta > 0$, such that if $|x-1| < \delta$, then $\left|x^2-1\right| < 1/10$.

28. Obtain a number, $\delta > 0$, such that if $|x-4| < \delta$, then $|\sqrt{x}-2| < 1/10$.

29. Suppose $\varepsilon > 0$ is a given positive number. Obtain a number, $\delta > 0$, such that if $|x-1| < \delta$, then $|\sqrt{x}-1| < \varepsilon$. **Hint:** This δ will depend in some way on ε. You need to tell how.

2.6 The Binomial Theorem

Consider the following problem: You have the integers $S_n = \{1, 2, \cdots, n\}$ and k is an integer no larger than n. How many ways are there to fill k slots with these integers starting from left to right if whenever an integer from S_n has been used, it cannot be re used in any succeeding slot?

$$\overbrace{\_\_\_\_, \_\_\_\_, \_\_\_\_, \_\_\_\_, \cdots, \_\_\_\_}^{k \text{ of these slots}}$$

This number is known as permutations of n things taken k at a time and is denoted by $P(n,k)$. It is easy to figure it out. There are n choices for the first slot. For each choice for the fist slot, there remain $n-1$ choices for the second slot. Thus there are $n(n-1)$ ways to fill the first two slots. Now there remain $n-2$ ways to fill the third. Thus there are $n(n-1)(n-2)$ ways to fill the first three slots. Continuing this way, you see there are

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

ways to do this.

Now define for k a positive integer,

$$k! \equiv k(k-1)(k-2)\cdots 1, \ 0! \equiv 1.$$

This is called k factorial. Thus $P(k,k) = k!$ and you should verify that

$$P(n,k) = \frac{n!}{(n-k)!}$$

Now consider the number of ways of selecting a set of k different numbers from S_n. For each set of k numbers there are $P(k,k) = k!$ ways of listing these numbers in order. Therefore, denoting by $\begin{pmatrix} n \\ k \end{pmatrix}$ the number of ways of selecting a set of k numbers from S_n, it must be the case that

$$\begin{pmatrix} n \\ k \end{pmatrix} k! = P(n,k) = \frac{n!}{(n-k)!}$$

Therefore,

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}.$$

How many ways are there to select no numbers from S_n? Obviously one way. Note the above formula gives the right answer in this case as well as in all other cases due to the definition which says $0! = 1$.

Now consider the problem of writing a formula for $(x+y)^n$ where n is a positive integer. Imagine writing it like this:

$$\overbrace{(x+y)(x+y)\cdots(x+y)}^{n \text{ times}}$$

Then you know the result will be sums of terms of the form $a_k x^k y^{n-k}$. What is a_k? In other words, how many ways can you pick x from k of the factors above and y from the other $n-k$. There are n factors so the number of ways to do it is

$$\begin{pmatrix} n \\ k \end{pmatrix}.$$

Therefore, a_k is the above formula and so this proves the following important theorem known as the binomial theorem.

Theorem 2.6.1 *The following formula holds for any n a positive integer.*

$$(x+y)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k y^{n-k}.$$

2.7 Well Ordering Principle And Archimedian Property

Definition 2.7.1 *A set is well ordered if every nonempty subset S, contains a smallest element z having the property that $z \le x$ for all $x \in S$.*

Axiom 2.7.2 *Any set of integers larger than a given number is well ordered.*

In particular, the natural numbers defined as

$$\mathbb{N} \equiv \{1, 2, \cdots\}$$

is well ordered.

The above axiom implies the principle of mathematical induction.

Theorem 2.7.3 *(Mathematical induction) A set $S \subseteq \mathbb{Z}$, having the property that $a \in S$ and $n+1 \in S$ whenever $n \in S$ contains all integers $x \in \mathbb{Z}$ such that $x \ge a$.*

Proof: Let $T \equiv ([a,\infty) \cap \mathbb{Z}) \setminus S$. Thus T consists of all integers larger than or equal to a which are not in S. The theorem will be proved if $T = \emptyset$. If $T \neq \emptyset$ then by the well ordering principle, there would have to exist a smallest element of T, denoted as b. It must be the case that $b > a$ since by definition, $a \notin T$. Then the integer, $b-1 \geq a$ and $b-1 \notin S$ because if $b-1 \in S$, then $b-1+1 = b \in S$ by the assumed property of S. Therefore, $b-1 \in ([a,\infty) \cap \mathbb{Z}) \setminus S = T$ which contradicts the choice of b as the smallest element of T. ($b-1$ is smaller.) Since a contradiction is obtained by assuming $T \neq \emptyset$, it must be the case that $T = \emptyset$ and this says that everything in $[a,\infty) \cap \mathbb{Z}$ is also in S.

Mathematical induction is a very useful device for proving theorems about the integers.

Example 2.7.4 *Prove by induction that* $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

By inspection, if $n = 1$ then the formula is true. The sum yields 1 and so does the formula on the right. Suppose this formula is valid for some $n \geq 1$ where n is an integer. Then

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 .\end{aligned}$$

The step going from the first to the second line is based on the assumption that the formula is true for n. This is called the induction hypothesis. Now simplify the expression in the second line,

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 .$$

This equals

$$(n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right)$$

and

$$\begin{aligned}\frac{n(2n+1)}{6} + (n+1) &= \frac{6(n+1) + 2n^2 + n}{6} \\ &= \frac{(n+2)(2n+3)}{6}\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},\end{aligned}$$

showing the formula holds for $n+1$ whenever it holds for n. This proves the formula by mathematical induction.

Example 2.7.5 *Show that for all* $n \in \mathbb{N}$, $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

If $n = 1$ this reduces to the statement that $\frac{1}{2} < \frac{1}{\sqrt{3}}$ which is obviously true. Suppose then that the inequality holds for n. Then

$$\begin{aligned}\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} &< \frac{1}{\sqrt{2n+1}} \frac{2n+1}{2n+2} \\ &= \frac{\sqrt{2n+1}}{2n+2}.\end{aligned}$$

The theorem will be proved if this last expression is less than $\frac{1}{\sqrt{2n+3}}$. This happens if and only if

$$\left(\frac{1}{\sqrt{2n+3}}\right)^2 = \frac{1}{2n+3} > \frac{2n+1}{(2n+2)^2}$$

which occurs if and only if $(2n+2)^2 > (2n+3)(2n+1)$ and this is clearly true which may be seen from expanding both sides. This proves the inequality.

Lets review the process just used. If S is the set of integers at least as large as 1 for which the formula holds, the first step was to show $1 \in S$ and then that whenever $n \in S$, it follows $n+1 \in S$. Therefore, by the principle of mathematical induction, S contains $[1, \infty) \cap \mathbb{Z}$, all positive integers. In doing an inductive proof of this sort, the set, S is normally not mentioned. One just verifies the steps above. First show the thing is true for some $a \in \mathbb{Z}$ and then verify that whenever it is true for m it follows it is also true for $m+1$. When this has been done, the theorem has been proved for all $m \geq a$.

Definition 2.7.6 *The Archimedian property states that whenever $x \in \mathbb{R}$, and $a > 0$, there exists $n \in \mathbb{N}$ such that $na > x$.*

Axiom 2.7.7 *$\mathbb{R}$ has the Archimedian property.*

This is not hard to believe. Just look at the number line. This Archimedian property is quite important because it shows every real number is smaller than some integer. It also can be used to verify a very important property of the rational numbers.

Theorem 2.7.8 *Suppose $x < y$ and $y - x > 1$. Then there exists an integer, $l \in \mathbb{Z}$, such that $x < l < y$. If x is an integer, there is no integer y satisfying $x < y < x + 1$.*

Proof: Let x be the smallest positive integer. Not surprisingly, $x = 1$ but this can be proved. If $x < 1$ then $x^2 < x$ contradicting the assertion that x is the smallest natural number. Therefore, 1 is the smallest natural number. This shows there is no integer, y, satisfying $x < y < x + 1$ since otherwise, you could subtract x and conclude $0 < y - x < 1$ for some integer $y - x$.

Now suppose $y - x > 1$ and let

$$S \equiv \{w \in \mathbb{N} : w \geq y\}.$$

The set S is nonempty by the Archimedian property. Let k be the smallest element of S. Therefore, $k - 1 < y$. Either $k - 1 \leq x$ or $k - 1 > x$. If $k - 1 \leq x$, then

$$y - x \leq y - (k-1) = \overbrace{y - k}^{\leq 0} + 1 \leq 1$$

contrary to the assumption that $y - x > 1$. Therefore, $x < k - 1 < y$ and this proves the theorem with $l = k - 1$.

It is the next theorem which gives the density of the rational numbers. This means that for any real number, there exists a rational number arbitrarily close to it.

Theorem 2.7.9 *If $x < y$ then there exists a rational number r such that $x < r < y$.*

Proof: Let $n \in \mathbb{N}$ be large enough that

$$n(y - x) > 1.$$

Thus $(y - x)$ added to itself n times is larger than 1. Therefore,

$$n(y - x) = ny + n(-x) = ny - nx > 1.$$

It follows from Theorem 2.7.8 there exists $m \in \mathbb{Z}$ such that

$$nx < m < ny$$

and so take $r = m/n$.

Definition 2.7.10 *A set, $S \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if whenever $a < b$, $S \cap (a, b) \neq \emptyset$.*

Thus the above theorem says $\mathbb{Q}$ is "dense" in $\mathbb{R}$.

You probably saw the process of division in elementary school. Even though you saw it at a young age it is very profound and quite difficult to understand. Suppose you want to do the following problem $\frac{79}{22}$. What did you do? You likely did a process of long division which gave the following result.

$$\frac{79}{22} = 3 \text{ with remainder } 13.$$

This meant

$$79 = 3(22) + 13.$$

You were given two numbers, 79 and 22 and you wrote the first as some multiple of the second added to a third number which was smaller than the second number. Can this always be done? The answer is in the next theorem and depends here on the Archimedian property of the real numbers.

Theorem 2.7.11 *Suppose $0 < a$ and let $b \geq 0$. Then there exists a unique integer p and real number r such that $0 \leq r < a$ and $b = pa + r$.*

Proof: Let $S \equiv \{n \in \mathbb{N} : an > b\}$. By the Archimedian property this set is nonempty. Let $p + 1$ be the smallest element of S. Then $pa \leq b$ because $p + 1$ is the smallest in S. Therefore,

$$r \equiv b - pa \geq 0.$$

If $r \geq a$ then $b - pa \geq a$ and so $b \geq (p + 1)a$ contradicting $p + 1 \in S$. Therefore, $r < a$ as desired.

To verify uniqueness of p and r, suppose p_i and r_i, $i = 1, 2$, both work and $r_2 > r_1$. Then a little algebra shows

$$p_1 - p_2 = \frac{r_2 - r_1}{a} \in (0, 1).$$

Thus $p_1 - p_2$ is an integer between 0 and 1, contradicting Theorem 2.7.8. The case that $r_1 > r_2$ cannot occur either by similar reasoning. Thus $r_1 = r_2$ and it follows that $p_1 = p_2$.

This theorem is called the Euclidean algorithm when a and b are integers.

2.8 Exercises

1. By Theorem 2.7.9 it follows that for $a < b$, there exists a rational number between a and b. Show there exists an integer k such that

$$a < \frac{k}{2^m} < b.$$

2. Show there is no smallest number in $(0,1)$. Recall $(0,1)$ means the real numbers which are strictly larger than 0 and smaller than 1.

3. Show there is no smallest number in $\mathbb{Q} \cap (0,1)$.

4. Show that if $S \subseteq \mathbb{R}$ and S is well ordered with respect to the usual order on $\mathbb{R}$ then S cannot be dense in $\mathbb{R}$.

5. Prove by induction that $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$.

6. It is a fine thing to be able to prove a theorem by induction but it is even better to be able to come up with a theorem to prove in the first place. Derive a formula for $\sum_{k=1}^{n} k^4$ in the following way. Look for a formula in the form $An^5 + Bn^4 + Cn^3 + Dn^2 + En + F$. Then try to find the constants A, B, C, D, E, and F such that things work out right. In doing this, show

$$(n+1)^4 =$$

$$\left(A(n+1)^5 + B(n+1)^4 + C(n+1)^3 + D(n+1)^2 + E(n+1) + F\right)$$

$$-An^5 + Bn^4 + Cn^3 + Dn^2 + En + F$$

and so some progress can be made by matching the coefficients. When you get your answer, prove it is valid by induction.

7. Prove by induction that whenever $n \geq 2$, $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$.

8. If $r \neq 0$, show by induction that $\sum_{k=1}^{n} ar^k = a\frac{r^{n+1}}{r-1} - a\frac{r}{r-1}$.

9. Prove by induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

10. Let a and d be real numbers. Find a formula for $\sum_{k=1}^{n} (a + kd)$ and then prove your result by induction.

11. Consider the geometric series, $\sum_{k=1}^{n} ar^{k-1}$. Prove by induction that if $r \neq 1$, then

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a - ar^n}{1 - r}.$$

12. This problem is a continuation of Problem 11. You put money in the bank and it accrues interest at the rate of r per payment period. These terms need a little explanation. If the payment period is one month, and you started with \$100 then the amount at the end of one month would equal $100(1+r) = 100 + 100r$. In this the second term is the interest and the first is called the principal. Now you have $100(1+r)$ in the bank. How much will you have at the end of the second month? By analogy to what was just done it would equal

$$100(1+r) + 100(1+r)r = 100(1+r)^2.$$

In general, the amount you would have at the end of n months would be $100(1+r)^n$. (When a bank says they offer 6% compounded monthly, this means r, the rate per payment period equals .06/12.) In general, suppose you start with P and it sits in the bank for n payment periods. Then at the end of the n^{th} payment period, you would have $P(1+r)^n$ in the bank. In an ordinary annuity, you make payments, P at the end of each payment period, the first payment at the end of the first payment period. Thus there are n payments in all. Each accrue interest at the rate of r per payment period. Using Problem 11, find a formula for the amount you will have in the bank at the end of n payment periods? This is called the future value of an ordinary annuity. **Hint:** The first payment sits in the bank for $n-1$ payment periods and so this payment becomes $P(1+r)^{n-1}$. The second sits in the bank for $n-2$ payment periods so it grows to $P(1+r)^{n-2}$, etc.

13. Now suppose you want to buy a house by making n equal monthly payments. Typically, n is pretty large, 360 for a thirty year loan. Clearly a payment made 10 years from now can't be considered as valuable to the bank as one made today. This is because the one made today could be invested by the bank and having accrued interest for 10 years would be far larger. So what is a payment made at the end of k payment periods worth today assuming money is worth r per payment period? Shouldn't it be the amount, Q which when invested at a rate of r per payment period would yield P at the end of k payment periods? Thus from Problem 12 $Q(1+r)^k = P$ and so $Q = P(1+r)^{-k}$. Thus this payment of P at the end of n payment periods, is worth $P(1+r)^{-k}$ to the bank right now. It follows the amount of the loan should equal the sum of these "discounted payments". That is, letting A be the amount of the loan,

$$A = \sum_{k=1}^{n} P(1+r)^{-k}.$$

Using Problem 11, find a formula for the right side of the above formula. This is called the present value of an ordinary annuity.

14. Suppose the available interest rate is 7% per year and you want to take a loan for \$100,000 with the first monthly payment at the end of the first month. If you want to pay off the loan in 20 years, what should the monthly payments be? **Hint:** The rate per payment period is .07/12. See the formula you got in Problem 13 and solve for P.

15. Consider the first five rows of Pascal's[2] triangle

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \end{array}$$

What is the sixth row? Now consider that $(x+y)^1 = 1x+1y$, $(x+y)^2 = x^2+2xy+y^2$, and $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$. Give a conjecture about that $(x+y)^5$.

16. Based on Problem 15 conjecture a formula for $(x+y)^n$ and prove your conjecture by induction. **Hint:** Letting the numbers of the n^{th} row of Pascal's triangle be denoted by $\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$ in reading from left to right, there is a relation between the numbers on the $(n+1)^{st}$ row and those on the n^{th} row, the relation being $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. This is used in the inductive step.

17. Let $\binom{n}{k} \equiv \frac{n!}{(n-k)!k!}$ where $0! \equiv 1$ and $(n+1)! \equiv (n+1)n!$ for all $n \geq 0$. Prove that whenever $k \geq 1$ and $k \leq n$, then $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. Are these numbers, $\binom{n}{k}$ the same as those obtained in Pascal's triangle? Prove your assertion.

[2] Blaise Pascal lived in the 1600's and is responsible for the beginnings of the study of probability.

18. The binomial theorem states $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. Prove the binomial theorem by induction. **Hint:** You might try using the preceding problem.

19. Show that for $p \in (0,1)$, $\sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k} = np$.

20. Using the binomial theorem prove that for all $n \in \mathbb{N}$, $\left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}$. **Hint:** Show first that $\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!}$. By the binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{\overbrace{n\cdot(n-1)\cdots(n-k+1)}^{k \text{ factors}}}{k!n^k}.$$

Now consider the term $\frac{n\cdot(n-1)\cdots(n-k+1)}{k!n^k}$ and note that a similar term occurs in the binomial expansion for $\left(1+\frac{1}{n+1}\right)^{n+1}$ except that n is replaced with $n+1$ whereever this occurs. Argue the term got bigger and then note that in the binomial expansion for $\left(1+\frac{1}{n+1}\right)^{n+1}$, there are more terms.

21. Prove by induction that for all $k \geq 4$, $2^k \leq k!$

22. Use the Problems 21 and 20 to verify for all $n \in \mathbb{N}$, $\left(1+\frac{1}{n}\right)^n \leq 3$.

23. Prove by induction that $1 + \sum_{i=1}^{n} i\,(i!) = (n+1)!$.

24. I can jump off the top of the Empire State Building without suffering any ill effects. Here is the proof by induction. If I jump from a height of one inch, I am unharmed. Furthermore, if I am unharmed from jumping from a height of n inches, then jumping from a height of $n+1$ inches will also not harm me. This is self evident and provides the induction step. Therefore, I can jump from a height of n inches for any n. What is the matter with this reasoning?

25. All horses are the same color. Here is the proof by induction. A single horse is the same color as himself. Now suppose the theorem that all horses are the same color is true for n horses and consider $n+1$ horses. Remove one of the horses and use the induction hypothesis to conclude the remaining n horses are all the same color. Put the horse which was removed back in and take out another horse. The remaining n horses are the same color by the induction hypothesis. Therefore, all $n+1$ horses are the same color as the $n-1$ horses which didn't get moved. This proves the theorem. Is there something wrong with this argument?

26. Let $\binom{n}{k_1, k_2, k_3}$ denote the number of ways of selecting a set of k_1 things, a set of k_2 things, and a set of k_3 things from a set of n things such that $\sum_{i=1}^{3} k_i = n$. Find a formula for $\binom{n}{k_1, k_2, k_3}$. Now give a formula for a trinomial theorem, one which expands $(x+y+z)^n$. Could you continue this way and get a multinomial formula?

2.9 Completeness of $\mathbb{R}$

By Theorem 2.7.9, between any two real numbers, points on the number line, there exists a rational number. This suggests there are a lot of rational numbers, but it is not clear from this Theorem whether the entire real line consists of only rational numbers. Some people might wish this were the case because then each real number could be described, not just as a point on a line but also algebraically, as the quotient of integers. Before 500 B.C., a group of mathematicians, led by Pythagoras believed in this, but they discovered their beliefs were false. It happened roughly like this. They knew they could construct the square root of two as the diagonal of a right triangle in which the two sides have unit length; thus they could regard $\sqrt{2}$ as a number. Unfortunately, they were also able to show $\sqrt{2}$ could not be written as the quotient of two integers. This discovery that the rational numbers could not even account for the results of geometric constructions was very upsetting to the Pythagoreans, especially when it became clear there were an endless supply of such "irrational" numbers.

This shows that if it is desired to consider all points on the number line, it is necessary to abandon the attempt to describe arbitrary real numbers in a purely algebraic manner using only the integers. Some might desire to throw out all the irrational numbers, and considering only the rational numbers, confine their attention to algebra, but this is not the approach to be followed here because it will effectively eliminate every major theorem of calculus. In this book real numbers will continue to be the points on the number line, a line which has no holes. This lack of holes is more precisely described in the following way.

Definition 2.9.1 *A non empty set, $S \subseteq \mathbb{R}$ is bounded above (below) if there exists $x \in \mathbb{R}$ such that $x \geq (\leq)\, s$ for all $s \in S$. If S is a nonempty set in $\mathbb{R}$ which is bounded above, then a number, l which has the property that l is an upper bound and that every other upper bound is no smaller than l is called a least upper bound, l.u.b.(S) or often $\sup(S)$. If S is a nonempty set bounded below, define the greatest lower bound, g.l.b.(S) or $\inf(S)$ similarly. Thus g is the g.l.b.(S) means g is a lower bound for S and it is the largest of all lower bounds. If S is a nonempty subset of $\mathbb{R}$ which is not bounded above, this information is expressed by saying $\sup(S) = +\infty$ and if S is not bounded below, $\inf(S) = -\infty$.*

Every existence theorem in calculus depends on some form of the completeness axiom.

Axiom 2.9.2 *(completeness) Every nonempty set of real numbers which is bounded above has a least upper bound and every nonempty set of real numbers which is bounded below has a greatest lower bound.*

It is this axiom which distinguishes Calculus from Algebra. A fundamental result about sup and inf is the following.

Proposition 2.9.3 *Let S be a nonempty set and suppose $\sup(S)$ exists. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ exists, then for every $\delta > 0$,

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

Proof: Consider the first claim. If the indicated set equals $\emptyset$, then $\sup(S) - \delta$ is an upper bound for S which is smaller than $\sup(S)$, contrary to the definition of $\sup(S)$ as the least upper bound. In the second claim, if the indicated set equals $\emptyset$, then $\inf(S) + \delta$ would be a lower bound which is larger than $\inf(S)$ contrary to the definition of $\inf(S)$.

2.10 Exercises

1. Let $S = [2, 5]$. Find $\sup S$. Now let $S = [2, 5)$. Find $\sup S$. Is $\sup S$ always a number in S? Give conditions under which $\sup S \in S$ and then give conditions under which $\inf S \in S$.

2. Show that if $S \neq \emptyset$ and is bounded above (below) then $\sup S$ ($\inf S$) is unique. That is, there is only one least upper bound and only one greatest lower bound. If $S = \emptyset$ can you conclude that 7 is an upper bound? Can you conclude 7 is a lower bound? What about 13.5? What about any other number?

3. Let S be a set which is bounded above and let $-S$ denote the set $\{-x : x \in S\}$. How are $\inf(-S)$ and $\sup(S)$ related? **Hint:** Draw some pictures on a number line. What about $\sup(-S)$ and $\inf S$ where S is a set which is bounded below?

4. Solve the following equations which involve absolute values.

   (a) $|x+1| = |2x+3|$

   (b) $|x+1| - |x+4| = 6$

5. Solve the following inequalities which involve absolute values.

   (a) $|2x-6| < 4$

   (b) $|x-2| < |2x+2|$

6. Which of the field axioms is being abused in the following argument that $0 = 2$? Let $x = y = 1$. Then
$$0 = x^2 - y^2 = (x-y)(x+y)$$
and so
$$0 = (x-y)(x+y).$$
Now divide both sides by $x - y$ to obtain
$$0 = x + y = 1 + 1 = 2.$$

7. Give conditions under which equality holds in the triangle inequality.

8. Let $k \leq n$ where k and n are natural numbers. $P(n,k)$, permutations of n things taken k at a time, is defined to be the number of different ways to form an ordered list of k of the numbers, $\{1, 2, \cdots, n\}$. Show
$$P(n,k) = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

9. Using the preceding problem, show the number of ways of selecting a set of k things from a set of n things is $\binom{n}{k}$.

10. Prove the binomial theorem from Problem 9. **Hint:** When you take $(x+y)^n$, note that the result will be a sum of terms of the form, $a_k x^{n-k} y^k$ and you need to determine what a_k should be. Imagine writing $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ where there are n factors in the product. Now consider what happens when you multiply. Each factor contributes either an x or a y to a typical term.

11. Prove by induction that $n < 2^n$ for all natural numbers, $n \geq 1$.

12. Prove by the binomial theorem and Problem 9 that the number of subsets of a given finite set containing n elements is 2^n.

13. Let n be a natural number and let $k_1 + k_2 + \cdots k_r = n$ where k_i is a non negative integer. The symbol
$$\binom{n}{k_1 k_2 \cdots k_r}$$
denotes the number of ways of selecting r subsets of $\{1, \cdots, n\}$ which contain $k_1, k_2 \cdots k_r$ elements in them. Find a formula for this number.

14. Is it ever the case that $(a+b)^n = a^n + b^n$ for a and b positive real numbers?

15. Is it ever the case that $\sqrt{a^2+b^2} = a+b$ for a and b positive real numbers?

16. Is it ever the case that $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$ for x and y positive real numbers?

17. Derive a formula for the multinomial expansion, $\left(\sum_{k=1}^{p} a_k\right)^n$ which is analogous to the binomial expansion. **Hint:** See Problem 10.

18. Suppose $a > 0$ and that x is a real number which satisfies the quadratic equation,
$$ax^2 + bx + c = 0.$$
Find a formula for x in terms of a and b and square roots of expressions involving these numbers. **Hint:** First divide by a to get
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$
Then add and subtract the quantity $b^2/4a^2$. Verify that
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2.$$
Now solve the result for x. The process by which this was accomplished in adding in the term $b^2/4a^2$ is referred to as completing the square. You should obtain the quadratic formula[3],
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
The expression $b^2 - 4ac$ is called the discriminant. When it is positive there are two different real roots. When it is zero, there is exactly one real root and when it equals a negative number there are no real roots.

19. Suppose $f(x) = 3x^2 + 7x - 17$. Find the value of x at which $f(x)$ is smallest by completing the square. Also determine $f(\mathbb{R})$ and sketch the graph of f. **Hint:**
$$\begin{aligned} f(x) &= 3\left(x^2 + \frac{7}{3}x - \frac{17}{3}\right) = 3\left(x^2 + \frac{7}{3}x + \frac{49}{36} - \frac{49}{36} - \frac{17}{3}\right) \\ &= 3\left(\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} - \frac{17}{3}\right). \end{aligned}$$

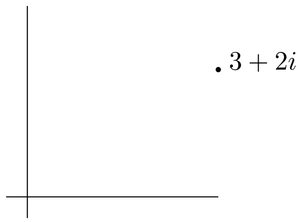
20. Suppose $f(x) = -5x^2 + 8x - 7$. Find $f(\mathbb{R})$. In particular, find the largest value of $f(x)$ and the value of x at which it occurs. Can you conjecture and prove a result about $y = ax^2 + bx + c$ in terms of the sign of a based on these last two problems?

[3] The ancient Babylonians knew how to solve these quadratic equations sometime before 1700 B.C. It seems they used pretty much the same process outlined in this exercise.

21. Show that if it is assumed $\mathbb{R}$ is complete, then the Archimedian property can be proved. **Hint:** Suppose completeness and let $a > 0$. If there exists $x \in \mathbb{R}$ such that $na \leq x$ for all $n \in \mathbb{N}$, then x/a is an upper bound for $\mathbb{N}$. Let l be the least upper bound and argue there exists $n \in \mathbb{N} \cap [l - 1/4, l]$. Now what about $n + 1$?

2.11 The Complex Numbers

Just as a real number should be considered as a point on the line, a complex number is considered a point in the plane which can be identified in the usual way using the Cartesian coordinates of the point. Thus (a, b) identifies a point whose x coordinate is a and whose y coordinate is b. In dealing with complex numbers, such a point is written as $a + ib$. For example, in the following picture, I have graphed the point $3 + 2i$. You see it corresponds to the point in the plane whose coordinates are $(3, 2)$.



Multiplication and addition are defined in the most obvious way subject to the convention that $i^2 = -1$. Thus,

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and

$$\begin{aligned}(a + ib)(c + id) &= ac + iad + ibc + i^2 bd \\ &= (ac - bd) + i(bc + ad).\end{aligned}$$

Every non zero complex number, $a + ib$, with $a^2 + b^2 \neq 0$, has a unique multiplicative inverse.

$$\frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

You should prove the following theorem.

Theorem 2.11.1 *The complex numbers with multiplication and addition defined as above form a field satisfying all the field axioms listed on Page 9.*

The field of complex numbers is denoted as $\mathbb{C}$. An important construction regarding complex numbers is the complex conjugate denoted by a horizontal line above the number. It is defined as follows.

$$\overline{a + ib} \equiv a - ib.$$

What it does is reflect a given complex number across the x axis. Algebraically, the following formula is easy to obtain.

$$\left(\overline{a + ib}\right)(a + ib) = a^2 + b^2.$$

Definition 2.11.2 *Define the absolute value of a complex number as follows.*

$$|a + ib| \equiv \sqrt{a^2 + b^2}.$$

Thus, denoting by z the complex number, $z = a + ib$,

$$|z| = (z\overline{z})^{1/2}.$$

With this definition, it is important to note the following. Be sure to verify this. It is not too hard but you need to do it.

Remark 2.11.3 *: Let $z = a + ib$ and $w = c + id$. Then $|z - w| = \sqrt{(a-c)^2 + (b-d)^2}$. Thus the distance between the point in the plane determined by the ordered pair, (a, b) and the ordered pair (c, d) equals $|z - w|$ where z and w are as just described.*

For example, consider the distance between $(2, 5)$ and $(1, 8)$. From the distance formula which you should have seen in either algebra of calculus, this distance is defined as

$$\sqrt{(2-1)^2 + (5-8)^2} = \sqrt{10}.$$

On the other hand, letting $z = 2 + i5$ and $w = 1 + i8$, $z - w = 1 - i3$ and so

$$(z - w)\left(\overline{z - w}\right) = (1 - i3)(1 + i3) = 10$$

so $|z - w| = \sqrt{10}$, the same thing obtained with the distance formula.

Notation 2.11.4 *From now on I will use the symbol $\mathbb{F}$ to denote either $\mathbb{C}$ or $\mathbb{R}$, rather than fussing over which one is meant because it often does not make any difference.*

The triangle inequality holds for the complex numbers just like it does for the real numbers.

Theorem 2.11.5 *Let $z, w \in \mathbb{C}$. Then*

$$|w + z| \leq |w| + |z|, \ ||z| - |w|| \leq |z - w|.$$

Proof: First note $|zw| = |z||w|$. Here is why: If $z = x + iy$ and $w = u + iv$, then

$$|zw|^2 = |(x + iy)(u + iv)|^2 = |xu - yv + i(xv + yu)|^2$$

$$= (xu - yv)^2 + (xv + yu)^2 = x^2u^2 + y^2v^2 + x^2v^2 + y^2u^2$$

Now look at the right side.

$$|z|^2 |w|^2 = (x + iy)(x - iy)(u + iv)(u - iv) = x^2u^2 + y^2v^2 + x^2v^2 + y^2u^2,$$

the same thing. Thus the rest of the proof goes just as before with real numbers. Using the results of Problem 6 on Page 33, the following holds.

$$\begin{aligned}
|z + w|^2 &= (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{\overline{w}z} \\
&= |z|^2 + |w|^2 + 2\operatorname{Re} z\overline{w} \\
&\leq |z|^2 + |w|^2 + 2|z\overline{w}| = |z|^2 + |w|^2 + 2|z||w| \\
&= (|z| + |w|)^2
\end{aligned}$$

and so $|z + w| \leq |z| + |w|$ as claimed. The other inequality follows as before.

$$|z| \leq |z - w| + |w|$$

and so

$$|z| - |w| \leq |z - w| = |w - z|.$$

Now do the same argument switching the roles of z and w to conclude

$$|z| - |w| \leq |z - w|, \ \ |w| - |z| \leq |z - w|$$

which implies the desired inequality. This proves the theorem.

2.12 Exercises

1. Let $z = 5 + i9$. Find z^{-1}.

2. Let $z = 2 + i7$ and let $w = 3 - i8$. Find $zw, z + w, z^2$, and w/z.

3. If z is a complex number, show there exists ω a complex number with $|\omega| = 1$ and $\omega z = |z|$.

4. For those who know about the trigonometric functions from calculus or trigonometry[4], De Moivre's theorem says
$$[r(\cos t + i \sin t)]^n = r^n (\cos nt + i \sin nt)$$
for n a positive integer. Prove this formula by induction. Does this formula continue to hold for all integers, n, even negative integers? Explain.

5. Using De Moivre's theorem from Problem 4, derive a formula for $\sin(5x)$ and one for $\cos(5x)$. **Hint:** Use Problem 18 on Page 27 and if you like, you might use Pascal's triangle to construct the binomial coefficients.

6. If z, w are complex numbers prove $\overline{zw} = \overline{z}\,\overline{w}$ and then show by induction that $\overline{z_1 \cdots z_m} = \overline{z_1} \cdots \overline{z_m}$. Also verify that $\overline{\sum_{k=1}^m z_k} = \sum_{k=1}^m \overline{z_k}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.

7. Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where all the a_k are real numbers. Suppose also that $p(z) = 0$ for some $z \in \mathbb{C}$. Show it follows that $p(\overline{z}) = 0$ also.

8. I claim that $1 = -1$. Here is why.
$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$
This is clearly a remarkable result but is there something wrong with it? If so, what is wrong?

9. De Moivre's theorem of Problem 4 is really a grand thing. I plan to use it now for rational exponents, not just integers.
$$1 = 1^{(1/4)} = (\cos 2\pi + i \sin 2\pi)^{1/4} = \cos(\pi/2) + i \sin(\pi/2) = i.$$
Therefore, squaring both sides it follows $1 = -1$ as in the previous problem. What does this tell you about De Moivre's theorem? Is there a profound difference between raising numbers to integer powers and raising numbers to non integer powers?

10. Review Problem 4 at this point. Now here is another question: If n is an integer, is it always true that $(\cos\theta - i \sin\theta)^n = \cos(n\theta) - i \sin(n\theta)$? Explain.

11. Suppose you have any polynomial in $\cos\theta$ and $\sin\theta$. By this I mean an expression of the form $\sum_{\alpha=0}^m \sum_{\beta=0}^n a_{\alpha\beta} \cos^\alpha \theta \sin^\beta \theta$ where $a_{\alpha\beta} \in \mathbb{C}$. Can this always be written in the form $\sum_{\gamma=-(n+m)}^{m+n} b_\gamma \cos\gamma\theta + \sum_{\tau=-(n+m)}^{n+m} c_\tau \sin\tau\theta$? Explain.

12. Does there exist a subset of $\mathbb{C}$, $\mathbb{C}^+$ which satisfies 2.1 - 2.3? **Hint:** You might review the theorem about order. Show -1 cannot be in $\mathbb{C}^+$. Now ask questions about $-i$ and i. In mathematics, you can sometimes show certain things do not exist. It is very seldom you can do this outside of mathematics. For example, does the Loch Ness monster exist? Can you prove it does not?

[4] I will present a treatment of the trig functions which is independent of plane geometry a little later.

Set Theory

3.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as 1,2,-4, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, $\{ \}$ specify a set by listing the things which are in the set between the parentheses. For example the set of integers between -1 and 2, including these numbers could be denoted as $\{-1, 0, 1, 2\}$. The notation signifying x is an element of a set S, is written as $x \in S$. Thus, $1 \in \{-1, 0, 1, 2, 3\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.
2. To every set, A, and to every condition $S(x)$ there corresponds a set, B, whose elements are exactly those elements x of A for which $S(x)$ holds.
3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.
4. The Cartesian product of a nonempty family of nonempty sets is nonempty.
5. If A is a set there exists a set, $\mathcal{P}(A)$ such that $\mathcal{P}(A)$ is the set of all subsets of A. This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1, 2, 3\} = \{2, 3, 1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some "condition" which can be used as a test to determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

$$\{x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z}\}.$$

In this notation, the colon is read as "such that" and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way to determine to everyone's satisfaction whether a given judge is an activist. Also, **just**

because something is grammatically correct does not mean it makes any sense. For example consider the following nonsense.

$$S = \{x \in \text{set of dogs } : \text{ it is colder in the mountains than in the winter}\}.$$

So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say "set of sets" or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If $\mathcal{S}$ is such a set whose elements are sets,

$$\cup \{A : A \in \mathcal{S}\} \text{ or } \cup \mathcal{S}$$

signify this union.

Something is in the Cartesian product of a set or "family" of sets if it consists of a single thing taken from each set in the family. Thus $(1,2,3) \in \{1,4,.2\} \times \{1,2,7\} \times \{4,3,7,9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathcal{S}$ is a set whose elements are sets,

$$\prod_{A \in \mathcal{S}} A$$

signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathcal{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

A is a subset of B, written $A \subseteq B$, if every element of A is also an element of B. This can also be written as $B \supseteq A$. A is a proper subset of B, written $A \subset B$ or $B \supset A$ if A is a subset of B but A is not equal to $B, A \neq B$. $A \cap B$ denotes the intersection of the two sets, A and B and it means the set of elements of A which are also elements of B. The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, A and B and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of A, denoted as A^C (or more precisely as $X \setminus A$) is a set obtained from using the axiom of specification to write

$$A^C \equiv \{x \in X : x \notin A\}$$

The symbol $\notin$ means: "is not an element of". Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as "all" or "there exists" are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7. Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical even though such usage may be grammatically correct. Quantifiers are used often enough that there are symbols for them. The symbol $\forall$ is read as "for all" or "for every" and the symbol $\exists$ is read as "there exists". Thus $\forall\forall\exists\exists$ could mean for every upside down A there exists a backwards E.

DeMorgan's laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, U. Then

$$\cup\left\{A^C : A \in \mathcal{S}\right\} = \left(\cap\left\{A : A \in \mathcal{S}\right\}\right)^C$$

and

$$\cap\left\{A^C : A \in \mathcal{S}\right\} = \left(\cup\left\{A : A \in \mathcal{S}\right\}\right)^C.$$

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathcal{S}$ be a set of sets then

$$B \cup \cup\left\{A : A \in \mathcal{S}\right\} = \cup\left\{B \cup A : A \in \mathcal{S}\right\}.$$

and: Let $\mathcal{S}$ be a set of sets show

$$B \cap \cup\left\{A : A \in \mathcal{S}\right\} = \cup\left\{B \cap A : A \in \mathcal{S}\right\}.$$

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it S. Then you could consider A the set of all elements of S which are not elements of themselves, this from the axiom of specification. If A is an element of itself, then it fails to qualify for inclusion in A. Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in A so it is an element of itself and so this can't be true either. Thus the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, "Nothing contains everything". Always beware of statements involving quantifiers wherever they occur, even this one. This little observation described above is due to Bertrand Russell and is called Russell's paradox.

3.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

Definition 3.2.1 *Let X and Y be sets.*

$$X \times Y \equiv \{(x, y) : x \in X \text{ and } y \in Y\}$$

A relation is defined to be a subset of $X \times Y$. A function, f, also called a mapping, is a relation which has the property that if (x, y) and (x, y_1) are both elements of the f, then $y = y_1$. The domain of f is defined as

$$D(f) \equiv \{x : (x, y) \in f\},$$

written as $f : D(f) \to Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, x and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, (x, y) which is an element of the function or mapping has an input, x and a unique output, y,denoted as $f(x)$ while the name of the function is f. "mapping" is often a noun meaning function. However, it also is a verb as in "f is mapping A to B ". That which a function is thought of as doing is also referred to using the word "maps" as in: f maps X to Y. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

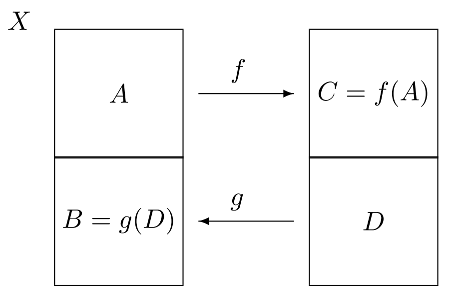
The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

Theorem 3.2.2 *Let $f : X \to Y$ and $g : Y \to X$ be two functions. Then there exist sets A, B, C, D, such that*

$$A \cup B = X,\ C \cup D = Y,\ A \cap B = \emptyset,\ C \cap D = \emptyset,$$

$$f(A) = C,\ g(D) = B.$$

The following picture illustrates the conclusion of this theorem.



Proof: Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \setminus f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if A, B, C, and D are as described above, A also would have this same property that the empty set has. However, A is probably larger. Therefore, say $A_0 \subseteq X$ satisfies $\mathcal{P}$ if whenever $y \in Y \setminus f(A_0)$, $g(y) \notin A_0$.

$$\mathcal{A} \equiv \{A_0 \subseteq X : A_0 \text{ satisfies } \mathcal{P}\}.$$

Let $A = \cup\mathcal{A}$. If $y \in Y \setminus f(A)$, then for each $A_0 \in \mathcal{A}$, $y \in Y \setminus f(A_0)$ and so $g(y) \notin A_0$. Since $g(y) \notin A_0$ for all $A_0 \in \mathcal{A}$, it follows $g(y) \notin A$. Hence A satisfies $\mathcal{P}$ and is the largest subset of X which does so. Now define

$$C \equiv f(A),\ D \equiv Y \setminus C,\ B \equiv X \setminus A.$$

It only remains to verify that $g(D) = B$.

Suppose $x \in B = X \setminus A$. Then $A \cup \{x\}$ does not satisfy $\mathcal{P}$ and so there exists $y \in Y \setminus f(A \cup \{x\}) \subseteq D$ such that $g(y) \in A \cup \{x\}$. But $y \notin f(A)$ and so since A satisfies $\mathcal{P}$, it follows $g(y) \notin A$. Hence $g(y) = x$ and so $x \in g(D)$ and this proves the theorem.

Theorem 3.2.3 *(Schroder Bernstein) If $f : X \to Y$ and $g : Y \to X$ are one to one, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: Let A, B, C, D be the sets of Theorem3.2.2 and define

$$h(x) \equiv \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in B \end{cases}$$

Then h is the desired one to one and onto mapping.

Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 3.2.4 *Let I be a set and let X_i be a set for each $i \in I$. f is a choice function written as*

$$f \in \prod_{i \in I} X_i$$

if $f(i) \in X_i$ for each $i \in I$.

The axiom of choice says that if $X_i \neq \emptyset$ for each $i \in I$, for I a set, then

$$\prod_{i \in I} X_i \neq \emptyset.$$

Sometimes the two functions, f and g are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 3.2.5 *If $f : X \to Y$ is onto and $g : Y \to X$ is onto, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: For each $y \in Y$, $f^{-1}(y) \equiv \{x \in X : f(x) = y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_0^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_0^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_0^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then f_0^{-1} is one to one because if $f_0^{-1}(y_1) = f_0^{-1}(y_2)$, then

$$y_1 = f\left(f_0^{-1}(y_1)\right) = f\left(f_0^{-1}(y_2)\right) = y_2.$$

Similarly g_0^{-1} is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h : X \to Y$ which is one to one and onto.

Definition 3.2.6 *A set S, is finite if there exists a natural number n and a map θ which maps $\{1, \cdots, n\}$ one to one and onto S. S is infinite if it is not finite. A set S, is called countable if there exists a map θ mapping $\mathbb{N}$ one to one and onto S.(When θ maps a set A to a set B, this will be written as $\theta : A \to B$ in the future.) Here $\mathbb{N} \equiv \{1, 2, \cdots\}$, the natural numbers. S is at most countable if there exists a map $\theta : \mathbb{N} \to S$ which is onto.*

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

Theorem 3.2.7 *If X and Y are both at most countable, then $X \times Y$ is also at most countable. If either X or Y is countable, then $X \times Y$ is also countable.*

Proof: It is given that there exists a mapping $\eta : \mathbb{N} \to X$ which is onto. Define $\eta(i) \equiv x_i$ and consider X as the set $\{x_1, x_2, x_3, \cdots\}$. Similarly, consider Y as the set $\{y_1, y_2, y_3, \cdots\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$\begin{array}{cccccc}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots & \leftarrow \text{Those which have } x_1 \text{ in first slot.} \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots & \leftarrow \text{Those which have } x_2 \text{ in first slot.} \\
(x_3, y_1) & (x_3, y_2) & (x_3, y_3) & \cdots & \leftarrow \text{Those which have } x_3 \text{ in first slot.} \cdot \\
\vdots & \vdots & \vdots & & \vdots
\end{array}$$

Follow a path through this array as follows.

$$\begin{array}{ccccccc}
(x_1, y_1) & \rightarrow & (x_1, y_2) & & (x_1, y_3) & \rightarrow \\
& \swarrow & & \nearrow & & \\
(x_2, y_1) & & (x_2, y_2) & & & \\
\downarrow & \nearrow & & & & \\
(x_3, y_1) & & & & &
\end{array}$$

Thus the first element of $X \times Y$ is (x_1, y_1), the second element of $X \times Y$ is (x_1, y_2), the third element of $X \times Y$ is (x_2, y_1) etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that X is countable. Then there exists $\alpha : \mathbb{N} \to X$ which is one to one and onto. Let $\beta : X \times Y \to \mathbb{N}$ be defined by $\beta((x, y)) \equiv \alpha^{-1}(x)$. Thus β is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 3.2.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$. This proves the theorem.

Theorem 3.2.8 *If X and Y are at most countable, then $X \cup Y$ is at most countable. If either X or Y are countable, then $X \cup Y$ is countable.*

Proof: As in the preceding theorem,

$$X = \{x_1, x_2, x_3, \cdots\}$$

and

$$Y = \{y_1, y_2, y_3, \cdots\}.$$

Consider the following array consisting of $X \cup Y$ and path through it.

$$\begin{array}{cccccc}
x_1 & \rightarrow & x_2 & & x_3 & \rightarrow \\
& \swarrow & & \nearrow & & \\
y_1 & \rightarrow & y_2 & & &
\end{array}$$

Thus the first element of $X \cup Y$ is x_1, the second is x_2 the third is y_1 the fourth is y_2 etc.

Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that X is countable and $\alpha : \mathbb{N} \to X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$,and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, β maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 3.2.5 yields the conclusion. This proves the theorem.

3.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 3.3.1 *Let S be a set. $\sim$ is an equivalence relation on S if it satisfies the following axioms.*

1. $x \sim x$ *for all* $x \in S$. *(Reflexive)*
2. *If* $x \sim y$ *then* $y \sim x$. *(Symmetric)*
3. *If* $x \sim y$ *and* $y \sim z$, *then* $x \sim z$. *(Transitive)*

Definition 3.3.2 *$[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x.*

With the above definition one can prove the following simple theorem.

Theorem 3.3.3 *Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.*

3.4 Exercises

1. The Barber of Seville is a man and he shaves exactly those men who do not shave themselves. Who shaves the Barber?

2. Do you believe each person who has ever lived on this earth has the right to do whatever he or she wants? (Note the use of the universal quantifier with no set in sight.) If you believe this, do you really believe what you say you believe? What of those people who want to deprive others their right to do what they want? Do people often use quantifiers this way?

3. President Bush, when he found there were no weapons of mass destruction said we would give the Iraqi's "freedom". He is protecting our "freedom". What is freedom? Is there an implied quantifier involved? Is there a set mentioned? What is the meaning of the statement? Could it mean different things to different people?

4. DeMorgan's laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, U. Show
$$\cup \left\{A^C : A \in \mathcal{S}\right\} = \left(\cap \left\{A : A \in \mathcal{S}\right\}\right)^C$$
and
$$\cap \left\{A^C : A \in \mathcal{S}\right\} = \left(\cup \left\{A : A \in \mathcal{S}\right\}\right)^C.$$

5. Let $\mathcal{S}$ be a set of sets show
$$B \cup \cup \left\{A : A \in \mathcal{S}\right\} = \cup \left\{B \cup A : A \in \mathcal{S}\right\}.$$

6. Let $\mathcal{S}$ be a set of sets show
$$B \cap \cup \left\{A : A \in \mathcal{S}\right\} = \cup \left\{B \cap A : A \in \mathcal{S}\right\}.$$

7. Show the rational numbers are countable, this is in spite of the fact that between any two integers there are infinitely many rational numbers. What does this show about the usefulness of common sense and instinct in mathematics?

8. We say a number is an algebraic number if it is the solution of an equation of the form

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

where all the a_j are integers and all exponents are also integers. Thus $\sqrt{2}$ is an algebraic number because it is a solution of the equation $x^2 - 2 = 0$. Using the observation that any such equation has at most n solutions, show the set of all algebraic numbers is countable.

9. Let A be a set and let $\mathcal{P}(A)$ be its power set, the set of all subsets of A. Show there does not exist any function f, which maps A onto $\mathcal{P}(A)$. Thus the power set is always strictly larger than the set from which it came. **Hint:** Suppose f is onto. Consider $S \equiv \{x \in A : x \notin f(x)\}$. If f is onto, then $f(y) = S$ for some $y \in A$. Is $y \in f(y)$? Note this argument holds for sets of any size.

10. The empty set is said to be a subset of every set. Why? Consider the statement: If pigs had wings, then they could fly. Is this statement true or false?

11. If $S = \{1, \cdots, n\}$, show $\mathcal{P}(S)$ has exactly 2^n elements in it. **Hint:** You might try a few cases first.

12. Show the set of all subsets of $\mathbb{N}$, the natural numbers, which have 3 elements, is countable. Is the set of all subsets of $\mathbb{N}$ which have finitely many elements countable? How about the set of all subsets of $\mathbb{N}$?

13. Prove Theorem 3.3.3.

14. Let S be a set and consider a function f which maps $\mathcal{P}(S)$ to $\mathcal{P}(S)$ which satisfies the following. If $A \subseteq B$, then $f(A) \subseteq f(B)$. Show there exists A such that $f(A) = A$. **Hint:** You might consider the following subset of $\mathcal{P}(S)$.

$$\mathcal{C} \equiv \{B \in \mathcal{P}(S) : B \subseteq f(B)\}$$

Then consider $A \equiv \cup\mathcal{C}$. Argue A is the "largest" set in $\mathcal{C}$ which implies A cannot be a proper subset of $f(A)$.

Functions And Sequences

4.1 General Considerations

The concept of a function is that of something which gives a unique output for a given input.

Definition 4.1.1 *Consider two sets, D and R along with a rule which assigns a unique element of R to every element of D. This rule is called a function and it is denoted by a letter such as f. The symbol, $D(f) = D$ is called the domain of f. The set R, also written $R(f)$, is called the range of f. The set of all elements of R which are of the form $f(x)$ for some $x \in D$ is often denoted by $f(D)$. When $R = f(D)$, the function, f, is said to be onto. It is common notation to write $f : D(f) \to R$ to denote the situation just described in this definition where f is a function defined on D having values in R.*

Example 4.1.2 *Consider the list of numbers, $\{1, 2, 3, 4, 5, 6, 7\} \equiv D$. Define a function which assigns an element of D to $R \equiv \{2, 3, 4, 5, 6, 7, 8\}$ by $f(x) \equiv x + 1$ for each $x \in D$.*

In this example there was a clearly defined procedure which determined the function. However, sometimes there is no discernible procedure which yields a particular function.

Example 4.1.3 *Consider the ordered pairs, $(1, 2), (2, -2), (8, 3), (7, 6)$ and let*

$$D \equiv \{1, 2, 8, 7\},$$

the set of first entries in the given set of ordered pairs, $R \equiv \{2, -2, 3, 6\}$, the set of second entries, and let $f(1) = 2, f(2) = -2, f(8) = 3,$ and $f(7) = 6$.

Sometimes functions are not given in terms of a formula. For example, consider the following function defined on the positive real numbers having the following definition.

Example 4.1.4 *For $x \in \mathbb{R}$ define*

$$f(x) = \begin{cases} \frac{1}{n} \text{ if } x = \frac{m}{n} \text{ in lowest terms for } m, n \in \mathbb{Z} \\ 0 \text{ if } x \text{ is not rational} \end{cases} \tag{4.1}$$

This is a very interesting function called the Dirichlet function. Note that it is not defined in a simple way from a formula.

Example 4.1.5 *Let D consist of the set of people who have lived on the earth except for Adam and for $d \in D$, let $f(d) \equiv$ the biological father of d. Then f is a function.*

This function is not the sort of thing studied in calculus but it is a function just the same. When $D(f)$ is not specified, it is understood to consist of everything for which f makes sense. The following definition gives several ways to make new functions from old ones.

Definition 4.1.6 *Let f, g be functions with values in $\mathbb{F}$. Let a, b be points of $\mathbb{F}$. Then $af + bg$ is the name of a function whose domain is $D(f) \cap D(g)$ which is defined as*

$$(af + bg)(x) = af(x) + bg(x).$$

The function, fg is the name of a function which is defined on $D(f) \cap D(g)$ given by

$$(fg)(x) = f(x)g(x).$$

Similarly for k an integer, f^k is the name of a function defined as

$$f^k(x) = (f(x))^k$$

The function, f/g is the name of a function whose domain is

$$D(f) \cap \{x \in D(g) : g(x) \neq 0\}$$

defined as

$$(f/g)(x) = f(x)/g(x).$$

If $f : D(f) \to X$ and $g : D(g) \to Y$, then $g \circ f$ is the name of a function whose domain is

$$\{x \in D(f) : f(x) \in D(g)\}$$

which is defined as

$$g \circ f(x) \equiv g(f(x)).$$

This is called the composition of the two functions.

You should note that $f(x)$ is not a function. It is the value of the function at the point, x. The name of the function is f. Nevertheless, people often write $f(x)$ to denote a function and it doesn't cause too many problems in beginning courses. When this is done, the variable, x should be considered as a generic variable free to be anything in $D(f)$.

Sometimes people get hung up on formulas and think that the only functions of importance are those which are given by some simple formula. It is a mistake to think this way. Functions involve a domain and a range and a function is determined by what it does.

Example 4.1.7 *Let $f(t) = t$ and $g(t) = 1 + t$. Then $fg : \mathbb{R} \to \mathbb{R}$ is given by*

$$fg(t) = t(1 + t) = t + t^2.$$

Example 4.1.8 *Let $f(t) = 2t + 1$ and $g(t) = \sqrt{1 + t}$. Then*

$$g \circ f(t) = \sqrt{1 + (2t + 1)} = \sqrt{2t + 2}$$

for $t \geq -1$. If $t < -1$ the inside of the square root sign is negative so makes no sense. Therefore, $g \circ f : \{t \in \mathbb{R} : t \geq -1\} \to \mathbb{R}$.

Note that in this last example, it was necessary to fuss about the domain of $g \circ f$ because g is only defined for certain values of t.

The concept of a one to one function is very important. This is discussed in the following definition.

Definition 4.1.9 *For any function, $f : D(f) \subseteq X \to Y$, define the following set known as the inverse image of y.*

$$f^{-1}(y) \equiv \{x \in D(f) : f(x) = y\}.$$

There may be many elements in this set, but when there is always only one element in this set for all $y \in f(D(f))$, the function f is one to one sometimes written, $1-1$. Thus f is one to one, $1-1$, if whenever $f(x) = f(x_1)$, then $x = x_1$. If f is one to one, the inverse function, f^{-1} is defined on $f(D(f))$ and $f^{-1}(y) = x$ where $f(x) = y$. Thus from the definition, $f^{-1}(f(x)) = x$ for all $x \in D(f)$ and $f\left(f^{-1}(y)\right) = y$ for all $y \in f(D(f))$. Defining id *by* $\text{id}(z) \equiv z$ *this says* $f \circ f^{-1} = \text{id}$ *and* $f^{-1} \circ f = \text{id}$.

Polynomials and rational functions are particularly easy functions to understand because they do come from a simple formula.

Definition 4.1.10 *A function f is a polynomial if*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i are real or complex numbers and n is a nonnegative integer. In this case the degree of the polynomial, $f(x)$ is n. Thus the degree of a polynomial is the largest exponent appearing on the variable.

f is a rational function if

$$f(x) = \frac{h(x)}{g(x)}$$

where h and g are polynomials.

For example, $f(x) = 3x^5 + 9x^2 + 7x + 5$ is a polynomial of degree 5 and

$$\frac{3x^5 + 9x^2 + 7x + 5}{x^4 + 3x + x + 1}$$

is a rational function.

Note that in the case of a rational function, the domain of the function might not be all of $\mathbb{F}$. For example, if

$$f(x) = \frac{x^2 + 8}{x + 1},$$

the domain of f would be all complex numbers not equal to -1.

Closely related to the definition of a function is the concept of the graph of a function.

Definition 4.1.11 *Given two sets, X and Y, the Cartesian product of the two sets, written as $X \times Y$, is assumed to be a set described as follows.*

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

$\mathbb{F}^2$ denotes the Cartesian product of $\mathbb{F}$ with $\mathbb{F}$. Recall $\mathbb{F}$ could be either $\mathbb{R}$ or $\mathbb{C}$.

The notion of Cartesian product is just an abstraction of the concept of identifying a point in the plane with an ordered pair of numbers.

Definition 4.1.12 *Let $f : D(f) \to R(f)$ be a function. The graph of f consists of the set,*

$$\{(x, y) : y = f(x) \text{ for } x \in D(f)\}.$$

Note that knowledge of the graph of a function is equivalent to knowledge of the function. To find $f(x)$, simply observe the ordered pair which has x as its first element and the value of y equals $f(x)$.

4.2 Sequences

Functions defined on the set of integers larger than a given integer are called sequences.

Definition 4.2.1 *A function whose domain is defined as a set of the form*

$$\{k, k+1, k+2, \cdots\}$$

for k an integer is known as a sequence. Thus you can consider $f(k), f(k+1), f(k+2)$, etc. Usually the domain of the sequence is either $\mathbb{N}$, the natural numbers consisting of $\{1, 2, 3, \cdots\}$ or the nonnegative integers, $\{0, 1, 2, 3, \cdots\}$. Also, it is traditional to write f_1, f_2, etc. instead of $f(1), f(2), f(3)$ etc. when referring to sequences. In the above context, f_k is called the first term, f_{k+1} the second and so forth. It is also common to write the sequence, not as f but as $\{f_i\}_{i=k}^{\infty}$ or just $\{f_i\}$ for short.

Example 4.2.2 *Let $\{a_k\}_{k=1}^{\infty}$ be defined by $a_k \equiv k^2 + 1$.*

This gives a sequence. In fact, $a_7 = a(7) = 7^2 + 1 = 50$ just from using the formula for the k^{th} term of the sequence.

It is nice when sequences come in this way from a formula for the k^{th} term. However, this is often not the case. Sometimes sequences are defined recursively. This happens, when the first several terms of the sequence are given and then a rule is specified which determines a_{n+1} from knowledge of $a_1, \cdots, a_n$. This rule which specifies a_{n+1} from knowledge of a_k for $k \leq n$ is known as a recurrence relation.

Example 4.2.3 *Let $a_1 = 1$ and $a_2 = 1$. Assuming $a_1, \cdots, a_{n+1}$ are known, $a_{n+2} \equiv a_n + a_{n+1}$.*

Thus the first several terms of this sequence, listed in order, are 1, 1, 2, 3, 5, 8,$\cdots$. This particular sequence is called the Fibonacci sequence and is important in the study of reproducing rabbits. Note this defines a function without giving a formula for it. Such sequences occur naturally in the solution of differential equations using power series methods and in many other situations of great importance.

For sequences, it is very important to consider something called a subsequence.

Definition 4.2.4 *Let $\{a_n\}$ be a sequence and let $n_1 < n_2 < n_3, \cdots$ be any strictly increasing list of integers such that n_1 is at least as large as the first number in the domain of the function. Then if $b_k \equiv a_{n_k}$, $\{b_k\}$ is called a subsequence of $\{a_n\}$.*

For example, suppose $a_n = \left(n^2 + 1\right)$. Thus $a_1 = 2$, $a_3 = 10$, etc. If

$$n_1 = 1, n_2 = 3, n_3 = 5, \cdots, n_k = 2k - 1,$$

then letting $b_k = a_{n_k}$, it follows

$$b_k = \left((2k-1)^2 + 1\right) = 4k^2 - 4k + 2.$$

4.3 Exercises

1. Let $g(t) \equiv \sqrt{2-t}$ and let $f(t) = \frac{1}{t}$. Find $g \circ f$. Include the domain of $g \circ f$. You can believe for now that the square root of any positive number exists. This will be shown later.

2. Give the domains of the following functions.

(a) $f(x) = \frac{x+3}{3x-2}$

(b) $f(x) = \sqrt{x^2-4}$

(c) $f(x) = \sqrt{4-x^2}$

(d) $f(x) = \sqrt{\frac{x\_4}{3x+5}}$

(e) $f(x) = \sqrt{\frac{x^2-4}{x+1}}$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) \equiv t^3 + 1$. Is f one to one? Can you find a formula for f^{-1}?

4. Suppose $a_1 = 1, a_2 = 3$, and $a_3 = -1$. Suppose also that for $n \geq 4$ it is known that $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$. Find a_7. Are you able to guess a formula for the k^{th} term of this sequence?

5. Let $f : \{t \in \mathbb{R} : t \neq -1\} \to \mathbb{R}$ be defined by $f(t) \equiv \frac{t}{t+1}$. Find f^{-1} if possible.

6. A function, $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function if whenever $x < y$, it follows that $f(x) < f(y)$. If f is a strictly increasing function, does f^{-1} always exist? Explain your answer.

7. Let $f(t)$ be defined by
$$f(t) = \begin{cases} 2t+1 \text{ if } t \leq 1 \\ t \text{ if } t > 1 \end{cases}.$$
Find f^{-1} if possible.

8. Suppose $f : D(f) \to R(f)$ is one to one, $R(f) \subseteq D(g)$, and $g : D(g) \to R(g)$ is one to one. Does it follow that $g \circ f$ is one to one?

9. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two one to one functions, which of the following are necessarily one to one on their domains? Explain why or why not by giving a proof or an example.

   (a) $f + g$

   (b) fg

   (c) f^3

   (d) f/g

10. Draw the graph of the function $f(x) = x^3 + 1$.

11. Draw the graph of the function $f(x) = x^2 + 2x + 2$.

12. Draw the graph of the function $f(x) = \frac{x}{1+x}$.

13. Suppose $a_n = \frac{1}{n}$ and let $n_k = 2^k$. Find b_k where $b_k = a_{n_k}$.

14. If X_i are sets and for some j, $X_j = \emptyset$, the empty set. Verify carefully that $\prod_{i=1}^n X_i = \emptyset$.

15. Suppose $f(x) + f\left(\frac{1}{x}\right) = 7x$ and f is a function defined on $\mathbb{R} \backslash \{0\}$, the nonzero real numbers. Find all values of x where $f(x) = 1$ if there are any. Does there exist any such function?

16. Does there exist a function f, satisfying $f(x) - f\left(\frac{1}{x}\right) = 3x$ which has both x and $\frac{1}{x}$ in the domain of f?

17. In the situation of the Fibonacci sequence show that the formula for the n^{th} term can be found and is given by

$$a_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Hint: You might be able to do this by induction but a better way would be to look for a solution to the recurrence relation, $a_{n+2} \equiv a_n + a_{n+1}$ of the form r^n. You will be able to show that there are two values of r which work, one of which is $r = \frac{1+\sqrt{5}}{2}$. Next you can observe that if r_1^n and r_2^n both satisfy the recurrence relation then so does $cr_1^n + dr_2^n$ for any choice of constants c, d. Then you try to pick c and d such that the conditions, $a_1 = 1$ and $a_2 = 1$ both hold.

18. In an ordinary annuity, you make constant payments, P at the beginning of each payment period. These accrue interest at the rate of r per payment period. This means at the start of the first payment period, there is the payment $P \equiv A_1$. Then this produces an amount rP in interest so at the beginning of the second payment period, you would have $rP + P + P \equiv A_2$. Thus $A_2 = A_1(1+r) + P$. Then at the beginning of the third payment period you would have $A_2(1+r)+P \equiv A_3$. Continuing in this way, you see that the amount in at the beginning of the n^{th} payment period would be A_n given by $A_n = A_{n-1}(1+r)+P$ and $A_1 = P$. Thus A is a function defined on the positive integers given recursively as just described and A_n is the amount at the beginning of the n^{th} payment period. Now if you wanted to find out A_n for large n, how would you do it? One way would be to use the recurrance relation n times. A better way would be to find a formula for A_n. Look for one in the form $A_n = Cz^n + s$ where C, z and s are to be determined. Show that $C = \frac{P}{r}, z = (1+r)$, and $s = -\frac{P}{r}$.

19. A well known puzzle consists of three pegs and several disks each of a different diameter, each having a hole in the center which allows it to be slid down each of the pegs. These disks are piled one on top of the other on one of the pegs, in order of decreasing diameter, the larger disks always being below the smaller disks. The problem is to move the whole pile of disks to another peg such that you never place a disk on a smaller disk. If you have n disks, how many moves will it take? Of course this depends on n. If $n = 1$, you can do it in one move. If $n = 2$, you would need 3. Let A_n be the number required for n disks. Then in solving the puzzle, you must first obtain the top $n-1$ disks arranged in order on another peg before you can move the bottom disk of the original pile. This takes A_{n-1} moves. Explain why $A_n = 2A_{n-1} + 1, A_1 = 1$ and give a formula for A_n. Look for one in the form $A_n = Cr^n + s$. This puzzle is called the Tower of Hanoi. When you have found a formula for A_n, explain why it is not possible to do this puzzle if n is very large.

4.4 The Limit Of A Sequence

The concept of the limit of a sequence was defined precisely by Bolzano.[1] The following is the precise definition of what is meant by the limit of a sequence.

[1] Bernhard Bolzano lived from 1781 to 1848. He was a Catholic priest and held a position in philosophy at the University of Prague. He had strong views about the absurdity of war, educational reform, and the need for individual concience. His convictions got him in trouble with Emporer Franz I of Austria and when he refused to recant, was forced out of the university. He understood the need for absolute rigor in mathematics. He also did work on physics.

Definition 4.4.1 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to* a,

$$\lim_{n\to\infty} a_n = a \text{ or } a_n \to a$$

if and only if for every $\varepsilon > 0$ *there exists* n_ε *such that whenever* $n \geq n_\varepsilon$,

$$|a_n - a| < \varepsilon.$$

Here a *and* a_n *are assumed to be complex numbers but the same definition holds more generally.*

In words the definition says that given any measure of closeness, ε, the terms of the sequence are eventually this close to a. Here, the word "eventually" refers to n being sufficiently large. The above definition is always the definition of what is meant by the limit of a sequence. If the a_n are complex numbers or later on, vectors the definition remains the same. If $a_n = x_n + iy_n$ and $a = x + iy$, $|a_n - a| = \sqrt{(x_n - x)^2 + (y_n - y)^2}$. Recall the way you measure distance between two complex numbers.

Theorem 4.4.2 *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} a_n = a_1$ *then* $a_1 = a$.

Proof: Suppose $a_1 \neq a$. Then let $0 < \varepsilon < |a_1 - a|/2$ in the definition of the limit. It follows there exists n_ε such that if $n \geq n_\varepsilon$, then $|a_n - a| < \varepsilon$ and $|a_n - a_1| < \varepsilon$. Therefore, for such n,

$$\begin{aligned} |a_1 - a| &\leq |a_1 - a_n| + |a_n - a| \\ &< \varepsilon + \varepsilon < |a_1 - a|/2 + |a_1 - a|/2 = |a_1 - a|, \end{aligned}$$

a contradiction.

Example 4.4.3 *Let* $a_n = \frac{1}{n^2+1}$.

Then it seems clear that

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$$

In fact, this is true from the definition. Let $\varepsilon > 0$ be given. Let $n_\varepsilon \geq \sqrt{\varepsilon^{-1}}$. Then if

$$n > n_\varepsilon \geq \sqrt{\varepsilon^{-1}},$$

it follows that $n^2 + 1 > \varepsilon^{-1}$ and so

$$0 < \frac{1}{n^2+1} = a_n < \varepsilon..$$

Thus $|a_n - 0| < \varepsilon$ whenever n is this large.

Note the definition was of no use in finding a candidate for the limit. This had to be produced based on other considerations. The definition is for verifying beyond any doubt that something is the limit. It is also what must be referred to in establishing theorems which are good for finding limits.

Example 4.4.4 *Let* $a_n = n^2$

Then in this case $\lim_{n\to\infty} a_n$ does not exist.

Example 4.4.5 *Let* $a_n = (-1)^n$.

In this case, $\lim_{n\to\infty}(-1)^n$ does not exist. This follows from the definition. Let $\varepsilon = 1/2$. If there exists a limit, l, then eventually, for all n large enough, $|a_n - l| < 1/2$. However, $|a_n - a_{n+1}| = 2$ and so,

$$2 = |a_n - a_{n+1}| \leq |a_n - l| + |l - a_{n+1}| < 1/2 + 1/2 = 1$$

which cannot hold. Therefore, there can be no limit for this sequence.

Theorem 4.4.6 *Suppose $\{a_n\}$ and $\{b_n\}$ are sequences and that*

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

Also suppose x and y are in $\mathbb{F}$. Then

$$\lim_{n\to\infty} xa_n + yb_n = xa + yb \tag{4.2}$$

$$\lim_{n\to\infty} a_n b_n = ab \tag{4.3}$$

If $b \neq 0$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}. \tag{4.4}$$

Proof: The first of these claims is left for you to do. To do the second, let $\varepsilon > 0$ be given and choose n_1 such that if $n \geq n_1$ then

$$|a_n - a| < 1.$$

Then for such n, the triangle inequality implies

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &\leq |a_n| |b_n - b| + |b| |a_n - a| \\ &\leq (|a| + 1) |b_n - b| + |b| |a_n - a| . \end{aligned}$$

Now let n_2 be large enough that for $n \geq n_2$,

$$|b_n - b| < \frac{\varepsilon}{2(|a| + 1)}, \text{ and } |a_n - a| < \frac{\varepsilon}{2(|b| + 1)}.$$

Such a number exists because of the definition of limit. Therefore, let

$$n_\varepsilon > \max(n_1, n_2).$$

For $n \geq n_\varepsilon$,

$$\begin{aligned} |a_n b_n - ab| &\leq (|a| + 1) |b_n - b| + |b| |a_n - a| \\ &< (|a| + 1) \frac{\varepsilon}{2(|a| + 1)} + |b| \frac{\varepsilon}{2(|b| + 1)} \leq \varepsilon. \end{aligned}$$

This proves 4.3. Next consider 4.4.

Let $\varepsilon > 0$ be given and let n_1 be so large that whenever $n \geq n_1$,

$$|b_n - b| < \frac{|b|}{2}.$$

Thus for such n,

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \left| \frac{a_n b - a b_n}{b b_n} \right| \leq \frac{2}{|b|^2} [|a_n b - ab| + |ab - ab_n|]$$

$$\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b|.$$

Now choose n_2 so large that if $n \geq n_2$, then

$$|a_n - a| < \frac{\varepsilon |b|}{4}, \text{ and } |b_n - b| < \frac{\varepsilon |b|^2}{4(|a|+1)}.$$

Letting $n_\varepsilon > \max(n_1, n_2)$, it follows that for $n \geq n_\varepsilon$,

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| \\ &< \frac{2}{|b|}\frac{\varepsilon |b|}{4} + \frac{2|a|}{|b|^2}\frac{\varepsilon |b|^2}{4(|a|+1)} < \varepsilon.\end{aligned}$$

Another very useful theorem for finding limits is the squeezing theorem.

Theorem 4.4.7 *Suppose* $\lim_{n\to\infty} a_n = a = \lim_{n\to\infty} b_n$ *and* $a_n \leq c_n \leq b_n$ *for all* n *large enough. Then* $\lim_{n\to\infty} c_n = a$.

Proof: Let $\varepsilon > 0$ be given and let n_1 be large enough that if $n \geq n_1$,

$$|a_n - a| < \varepsilon/2 \text{ and } |b_n - a| < \varepsilon/2.$$

Then for such n,

$$|c_n - a| \leq |a_n - a| + |b_n - a| < \varepsilon.$$

The reason for this is that if $c_n \geq a$, then

$$|c_n - a| = c_n - a \leq b_n - a \leq |a_n - a| + |b_n - a|$$

because $b_n \geq c_n$. On the other hand, if $c_n \leq a$, then

$$|c_n - a| = a - c_n \leq a - a_n \leq |a - a_n| + |b - b_n|.$$

This proves the theorem.

As an example, consider the following.

Example 4.4.8 *Let*

$$c_n \equiv (-1)^n \frac{1}{n}$$

and let $b_n = \frac{1}{n}$, *and* $a_n = -\frac{1}{n}$. *Then you may easily show that*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Since $a_n \leq c_n \leq b_n$, *it follows* $\lim_{n\to\infty} c_n = 0$ *also.*

Theorem 4.4.9 $\lim_{n\to\infty} r^n = 0$. *Whenever* $|r| < 1$.

Proof: If $0 < r < 1$ if follows $r^{-1} > 1$. Why? Letting $\alpha = \frac{1}{r} - 1$, it follows

$$r = \frac{1}{1+\alpha}.$$

Therefore, by the binomial theorem,

$$0 < r^n = \frac{1}{(1+\alpha)^n} \leq \frac{1}{1+\alpha n}.$$

Therefore, $\lim_{n\to\infty} r^n = 0$ if $0 < r < 1$. Now in general, if $|r| < 1$, $|r^n| = |r|^n \to 0$ by the first part. This proves the theorem.

An important theorem is the one which states that if a sequence converges, so does every subsequence. You should review Definition 4.2.4 on Page 46 at this point.

Theorem 4.4.10 *Let $\{x_n\}$ be a sequence with $\lim_{n\to\infty} x_n = x$ and let $\{x_{n_k}\}$ be a subsequence. Then $\lim_{k\to\infty} x_{n_k} = x$.*

Proof: Let $\varepsilon > 0$ be given. Then there exists n_ε such that if $n > n_\varepsilon$, then $|x_n - x| < \varepsilon$. Suppose $k > n_\varepsilon$. Then $n_k \geq k > n_\varepsilon$ and so

$$|x_{n_k} - x| < \varepsilon$$

showing $\lim_{k\to\infty} x_{n_k} = x$ as claimed.

Theorem 4.4.11 *Let $\{x_n\}$ be a sequence of real numbers and suppose each $x_n \leq l$ $(\geq l)$ and $\lim_{n\to\infty} x_n = x$. Then $x \leq l\,(\geq l)$. More generally, suppose $\{x_n\}$ and $\{y_n\}$ are two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then if $x_n \leq y_n$ for all n sufficiently large, then $x \leq y$.*

Proof: Let $\varepsilon > 0$ be given. Then for n large enough,

$$l \geq x_n > x - \varepsilon$$

and so

$$l + \varepsilon \geq x.$$

Since $\varepsilon > 0$ is arbitrary, this requires $l \geq x$. The other case is entirely similar or else you could consider $-l$ and $\{-x_n\}$ and apply the case just considered.

Consider the last claim. There exists N such that if $n \geq N$ then $x_n \leq y_n$ and

$$|x - x_n| + |y - y_n| < \varepsilon/2.$$

Then considering $n > N$ in what follows,

$$x - y \leq x_n + \varepsilon/2 - (y_n - \varepsilon/2) = x_n - y_n + \varepsilon \leq \varepsilon.$$

Since ε was arbitrary, it follows

$$x - y \leq 0.$$

This proves the theorem.

4.5 The Nested Interval Lemma

In Russia there is a kind of doll called a matrushka doll. You pick it up and notice it comes apart in the center. Separating the two halves you find an identical doll inside. Then you notice this inside doll also comes apart in the center. Separating the two halves, you find yet another identical doll inside. This goes on quite a while until the final doll is in one piece. The nested interval lemma is like a matrushka doll except the process never stops. It involves a sequence of intervals, the first containing the second, the second containing the third, the third containing the fourth and so on. The fundamental question is whether there exists a point in all the intervals. Sometimes there is such a point and this comes from completeness.

Lemma 4.5.1 *Let* $I_k = \left[a^k, b^k\right]$ *and suppose that for all* $k = 1, 2, \cdots$,

$$I_k \supseteq I_{k+1}.$$

Then there exists a point, $c \in \mathbb{R}$ *which is an element of every* I_k.

Proof: Since $I_k \supseteq I_{k+1}$, this implies

$$a^k \leq a^{k+1},\ b^k \geq b^{k+1}. \tag{4.5}$$

Consequently, if $k \leq l$,

$$a^l \leq a^l \leq b^l \leq b^k. \tag{4.6}$$

Now define

$$c \equiv \sup\left\{a^l : l = 1, 2, \cdots\right\}$$

By the first inequality in 4.5, and 4.6

$$a^k \leq c = \sup\left\{a^l : l = k, k+1, \cdots\right\} \leq b^k \tag{4.7}$$

for each $k = 1, 2 \cdots$. Thus $c \in I_k$ for every k and this proves the lemma. If this went too fast, the reason for the last inequality in 4.7 is that from 4.6, b^k is an upper bound to $\left\{a^l : l = k, k+1, \cdots\right\}$. Therefore, it is at least as large as the least upper bound.

This is really quite a remarkable result and may not seem so obvious. Consider the intervals $I_k \equiv (0, 1/k)$. Then there is no point which lies in all these intervals because no negative number can be in all the intervals and $1/k$ is smaller than a given positive number whenever k is large enough. Thus the only candidate for being in all the intervals is 0 and 0 has been left out of them all. The problem here is that the endpoints of the intervals were not included, contrary to the hypotheses of the above lemma in which all the intervals included the endpoints.

4.6 Exercises

1. Find $\lim_{n\to\infty} \frac{n}{3n+4}$.

2. Find $\lim_{n\to\infty} \frac{3n^4+7n+1000}{n^4+1}$.

3. Find $\lim_{n\to\infty} \frac{2^n+7(5^n)}{4^n+2(5^n)}$.

4. Find $\lim_{n\to\infty} \sqrt{(n^2+6n)} - n$. **Hint:** Multiply and divide by $\sqrt{(n^2+6n)} + n$.

5. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^k}$.

6. Suppose $\{x_n + iy_n\}$ is a sequence of complex numbers which converges to the complex number $x + iy$. Show this happens if and only if $x_n \to x$ and $y_n \to y$.

7. For $|r| < 1$, find $\lim_{n\to\infty} \sum_{k=0}^{n} r^k$. **Hint:** First show $\sum_{k=0}^{n} r^k = \frac{r^{n+1}}{r-1} - \frac{1}{r-1}$. Then recall Theorem 4.4.9.

8. Using the binomial theorem prove that for all $n \in \mathbb{N}$, $\left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1}$. **Hint:** Show first that $\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{k!}$. By the binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n}\frac{\overbrace{n\cdot(n-1)\cdots(n-k+1)}^{k \text{ factors}}}{k!n^k}.$$

Now consider the term $\frac{n\cdot(n-1)\cdots(n-k+1)}{k!n^k}$ and note that a similar term occurs in the binomial expansion for $\left(1+\frac{1}{n+1}\right)^{n+1}$ except you replace n with $n+1$ whereever this occurs. Argue the term got bigger and then note that in the binomial expansion for $\left(1+\frac{1}{n+1}\right)^{n+1}$, there are more terms.

9. Prove by induction that for all $k \geq 4$, $2^k \leq k!$

10. Use the Problems 21 on Page 27 and 8 to verify for all $n \in \mathbb{N}$, $\left(1+\frac{1}{n}\right)^n \leq 3$.

11. Prove $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and equals a number less than 3.

12. Using Problem 10, prove $n^{n+1} \geq (n+1)^n$ for all integers, $n \geq 3$.

13. Find $\lim_{n\to\infty} n\sin n$ if it exists. If it does not exist, explain why it does not.

14. Recall the axiom of completeness states that a set which is bounded above has a least upper bound and a set which is bounded below has a greatest lower bound. Show that a monotone decreasing sequence which is bounded below converges to its greatest lower bound. **Hint:** Let a denote the greatest lower bound and recall that because of this, it follows that for all $\varepsilon > 0$ there exist points of $\{a_n\}$ in $[a, a+\varepsilon]$.

15. Let $A_n = \sum_{k=2}^{n} \frac{1}{k(k-1)}$ for $n \geq 2$. Show $\lim_{n\to\infty} A_n$ exists and find the limit. **Hint:** Show there exists an upper bound to the A_n as follows.

$$\begin{aligned} \sum_{k=2}^{n} \frac{1}{k(k-1)} &= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ &= \frac{1}{2} - \frac{1}{n-1} \leq \frac{1}{2}. \end{aligned}$$

16. Let $H_n = \sum_{k=1}^{n} \frac{1}{k^2}$ for $n \geq 2$. Show $\lim_{n\to\infty} H_n$ exists. **Hint:** Use the above problem to obtain the existence of an upper bound.

17. Let $I_n = (-1/n, 1/n)$ and let $J_n = (0, 2/n)$. The intervals, I_n and J_n are open intervals of length $2/n$. Find $\cap_{n=1}^{\infty} I_n$ and $\cap_{n=1}^{\infty} J_n$. Repeat the same problem for $I_n = (-1/n, 1/n]$ and $J_n = [0, 2/n)$.

4.7 Sequential Compactness

4.7.1 Sequential Compactness

First I will discuss the very important concept of sequential compactness. This is a property that some sets have. A set of numbers is sequentially compact if every sequence contained in the set has a subsequence which converges to a point in the set. It is unbelievably useful whenever you try to understand existence theorems.

Definition 4.7.1 *A set, $K \subseteq \mathbb{F}$ is sequentially compact if whenever $\{a_n\} \subseteq K$ is a sequence, there exists a subsequence, $\{a_{n_k}\}$ such that this subsequence converges to a point of K.*

The following theorem is part of the Heine Borel theorem.

Theorem 4.7.2 *Every closed interval, $[a, b]$ is sequentially compact.*

Proof: Let $\{x_n\} \subseteq [a,b] \equiv I_0$. Consider the two intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ each of which has length $(b-a)/2$. At least one of these intervals contains x_n for infinitely many values of n. Call this interval I_1. Now do for I_1 what was done for I_0. Split it in half and let I_2 be the interval which contains x_n for infinitely many values of n. Continue this way obtaining a sequence of nested intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots$ where the length of I_n is $(b-a)/2^n$. Now pick n_1 such that $x_{n_1} \in I_1$, n_2 such that $n_2 > n_1$ and $x_{n_2} \in I_2, n_3$ such that $n_3 > n_2$ and $x_{n_3} \in I_3$, etc. (This can be done because in each case the intervals contained x_n for infinitely many values of n.) By the nested interval lemma there exists a point, c contained in all these intervals. Furthermore,

$$|x_{n_k} - c| < (b-a) 2^{-k}$$

and so $\lim_{k\to\infty} x_{n_k} = c \in [a,b]$. This proves the theorem.

4.7.2 Closed And Open Sets

I have been using the terminology $[a,b]$ is a closed interval to mean it is an interval which contains the two endpoints. However, there is a more general notion of what it means to be closed. Similarly there is a general notion of what it means to be open.

Definition 4.7.3 *Let U be a set of points. A point, $p \in U$ is said to be an interior point if whenever $|x-p|$ is sufficiently small, it follows $x \in U$ also. The set of points, x which are closer to p than δ is denoted by*

$$B(p,\delta) \equiv \{x \in \mathbb{F} : |x-p| < \delta\}.$$

This symbol, $B(p,\delta)$ is called an open ball of radius δ. Thus a point, p is an interior point of U if there exists $\delta > 0$ such that $p \in B(p,\delta) \subseteq U$. An open set is one for which every point of the set is an interior point. Closed sets are those which are complements of open sets. Thus H is closed means H^C is open.

Definition 4.7.4 *Let A be any nonempty set and let x be a point. Then x is said to be a limit point of A if for every $r > 0, B(x,r)$ contains a point of A which is not equal to x.*

Example 4.7.5 *Consider $A = \mathbb{N}$, the positive integers. Then none of the points of A is a limit point of A because if $n \in A, B(n, 1/10)$ contains no points of $\mathbb{N}$ which are not equal to n.*

Example 4.7.6 *Consider $A = (a,b)$, an open interval. If $x \in (a,b)$, let*

$$r = \min(|x-a|, |x-b|).$$

Then $B(x,r) \subseteq A$ because if $|y-x| < r$, then

$$\begin{aligned} y - a &= y - x + x - a \geq x - a - |y-x| \\ &= |x-a| - |y-x| > |x-a| - r \geq 0 \end{aligned}$$

showing $y > a$. A similar argument which you should provide shows $y < b$. Thus $y \in (a,b)$ and x is an interior point. Since x was arbitrary, this shows every point of (a,b) is an interior point and so (a,b) is open.

Theorem 4.7.7 *Let A be a nonempty set. A point a is a limit point of A if and only if there exists a sequence of distinct points of A, $\{a_n\}$ which converges to a. Also a nonempty set, A is closed if and only if it contains all its limit points.*

Proof: Suppose first it is a limit point of A. There exists $a_1 \in B(a,1) \cap A$ such that $a_1 \neq a$. Now supposing distinct points, $a_1, \cdots, a_n$ have been chosen such that none are equal to a and for each $k \leq n$, $a_k \in B(a, 1/k)$, let

$$0 < r_{n+1} < \min\left\{\frac{1}{n+1}, |a-a_1|, \cdots, |a-a_n|\right\}.$$

Then there exists $a_{n+1} \in B(a, r_{n+1}) \cap A$ with $a_{n+1} \neq a$. Because of the definition of r_{n+1}, a_{n+1} is not equal to any of the other a_k for $k < n+1$. Also since $|a - a_m| < 1/m$, it follows $\lim_{m\to\infty} a_m = a$. Conversely, if there exists a sequence of distinct points of A converging to a, then $B(a,r)$ contains all a_n for n large enough. Thus $B(a,r)$ contains infinitely many points of A since all are distinct. Thus at least one of them is not equal to a. This establishes the first part of the theorem.

Now consider the second claim. If A is closed then it is the complement of an open set. Since A^C is open, it follows that if $a \in A^C$, then there exists $\delta > 0$ such that $B(a,\delta) \subseteq A^C$ and so no point of A^C can be a limit point of A. In other words, every limit point of A must be in A. Conversely, suppose A contains all its limit points. Then A^C does not contain any limit points of A. It also contains no points of A. Therefore, if $a \in A^C$, since it is not a limit point of A, there exists $\delta > 0$ such that $B(a,\delta)$ contains no points of A different than a. However, a itself is not in A because $a \in A^C$. Therefore, $B(a,\delta)$ is entirely contained in A^C. Since $a \in A^C$ was arbitrary, this shows every point of A^C is an interior point and so A^C is open. This proves the theorem.

Theorem 4.7.8 *If K is sequentially compact and if H is a closed subset of K then H is sequentially compact.*

Proof: Let $\{x_n\} \subseteq H$. Then since K is sequentially compact, there is a subsequence, $\{x_{n_k}\}$ which converges to a point, $x \in K$. If $x \notin H$, then by Theorem 4.7.7, which says H^C is open, it follows there exists $B(x,r)$ such that this open ball contains no points of H. However, this is a contradiction to having $x_{n_k} \to x$ which requires $x_{n_k} \in B(x,r)$ for all k large enough. Thus $x \in H$ and this has shown H is sequentially compact.

Thus every closed subset of a closed interval is sequentially compact. This is equivalent to the following corollary.

Corollary 4.7.9 *Every closed and bounded set in $\mathbb{R}$ is sequentially compact.*

Proof: Let H be a closed and bounded set in $\mathbb{R}$. Then H is a closed subset of some interval of the form $[a,b]$. Therefore, it is sequentially compact.

What about the sequentially compact sets in $\mathbb{C}$?

Definition 4.7.10 *A set $S \subseteq \mathbb{C}$ is bounded if there is some $r > 0$ such that $S \subseteq B(0,r)$.*

Theorem 4.7.11 *Every closed and bounded set in $\mathbb{C}$ is sequentially compact.*

Proof: Let H be a closed and bounded set in $\mathbb{C}$. Then $H \subseteq B(0,r)$ for some r. Therefore,

$$H \subseteq \{x+iy : x \in [-r,r] \text{ and } y \in [-r,r]\} \equiv Q$$

because if $x+iy \in B(0,r)$, then $\sqrt{x^2+y^2} \leq r$ and so both $|x|, |y| \leq r$ which is the same as saying $x \in [-r,r]$ and $y \in [-r,r]$. Now let $\{x_n + iy_n\}_{n=1}^{\infty}$ be a sequence of points in H. Then $\{x_n\}$ is a sequence of points in $[-r,r]$ and $\{y_n\}$ is a sequence of points in $[-r,r]$. It follows from Theorem 4.7.2 there exists a subsequence of $\{x_n\}, \{x_{n_k}\}$ which converges to a point x in $[-r,r]$. Then $\{y_{n_k}\}$ is a sequence of points in $[-r,r]$ and so it has a subsequence, $\left\{y_{n_{k_l}}\right\}_{l=1}^{\infty}$ which converges to a point $y \in [-r,r]$. Thus $\left\{x_{n_{k_l}}\right\}_{l=1}^{\infty}$ converges to $x \in [-r,r]$ by Theorem 4.4.10 and as just noted, $\left\{y_{n_{k_l}}\right\}_{l=1}^{\infty}$ converges to $y \in [-r,r]$. It follows from the definition of distance in $\mathbb{C}$ that

$$x_{n_{k_l}} + iy_{n_{k_l}} \to x + iy \in Q.$$

However, H is closed and so $x + iy \in H$. This proves the theorem.

What are some examples of closed and bounded sets in $\mathbb{F}$?

Proposition 4.7.12 *Let $D(z,r)$ denote the set of points,*

$$\{w \in \mathbb{F} : |w - z| \leq r\}$$

Then $D(z,r)$ is closed and bounded. Also any set of the form

$$[a,b] + i[c,d]$$

is closed and bounded. Thus sets $D(z,r)$ and $[a,b] + i[c,d]$ are sequentially compact.

Proof: First note the set is bounded because

$$D(z,r) \subseteq B(0, |z| + 2r)$$

Here is why. Let $x \in D(z,r)$. Then $|x - z| \leq r$ and so

$$|x| \leq |x - z| + |z| \leq r + |z| < 2r + |z|.$$

It remains to verify it is closed. Suppose then that $y \notin D(z,r)$. This means $|y - z| > r$. Consider the open ball $B(y, |y - z| - r)$. If $x \in B(y, |y - z| - r)$, then

$$|x - y| < |y - z| - r$$

and so by the triangle inequality,

$$|z - x| \geq |z - y| - |y - x| > |x - y| + r - |x - y| = r$$

Thus the complement of $D(z,r)$ is open and so $D(z,r)$ is closed.

For the second type of set, if $x + iy \in [a,b] + i[c,d]$, then

$$|x + iy|^2 = x^2 + y^2 \leq (|a| + |b|)^2 + (|c| + |d|)^2 \equiv r/2.$$

Then $[a,b] + i[c,d] \subseteq B(0,r)$. Thus the set is bounded. It remains to verify it is closed. Suppose then that $x + iy \notin [a,b] + i[c,d]$. Then either $x \notin [a,b]$ or $y \notin [c,d]$. Say $y \notin [c,d]$. Then either $y > d$ or $y < c$. Say $y > d$. Consider

$$B(x + iy, y - d).$$

If $u + iv \in B(x + iy, y - d)$, then

$$\sqrt{(x - u)^2 + (y - v)^2} < y - d.$$

In particular, $|y - v| < y - d$ and so $v - y > d - y$ so $v > d$. Thus $u + iv \notin [a,b] + i[c,d]$ which shows the complement of $[a,b] + i[c,d]$ is open. The other cases are similar. This proves the proposition.

4.8 Exercises

1. Show the intersection of any collection of closed sets is closed and the union of any collection of open sets is open.

2. Show that if H is closed and U is open, then $H \setminus U$ is closed. Next show that $U \setminus H$ is open.

3. Show the finite intersection of any collection of open sets is open.

4. Show the finite union of any collection of closed sets is closed.

5. Suppose $\{H_n\}_{n=1}^N$ is a finite collection of sets and suppose x is a limit point of $\cup_{n=1}^N H_n$. Show x must be a limit point of at least one H_n.

6. Give an example of a set of closed sets whose union is not closed.

7. Give an example of a set of open sets whose intersection is not open.

8. Give an example of a set of closed sets whose union is open.

9. Give an example of a set of open sets whose intersection is closed.

10. Explain why $\mathbb{F}$ and $\emptyset$ are sets which are both open and closed when considered as subsets of $\mathbb{F}$.

11. Let A be a nonempty set of points and let A' denote the set of limit points of A. Show $A \cup A'$ is closed. **Hint:** You must show the limit points of $A \cup A'$ are in $A \cup A'$.

12. Let U be any open set in $\mathbb{F}$. Show that every point of U is a limit point of U.

13. Suppose $\{K_n\}$ is a sequence of sequentially compact nonempty sets which have the property that $K_n \supseteq K_{n+1}$ for all n. Show there exists a point in the intersection of all these sets, denoted by $\cap_{n=1}^\infty K_n$.

14. Now suppose $\{K_n\}$ is a sequence of sequentially compact nonempty sets which have the finite intersection property, every finite subset of $\{K_n\}$ has nonempty intersection. Show there exists a point in $\cap_{n=1}^\infty K_n$.

15. Start with the unit interval, $I_0 \equiv [0,1]$. Delete the middle third open interval, $(1/3, 2/3)$ resulting in the two closed intervals, $I_1 = [0, 1/3] \cup [2/3, 1]$. Next delete the middle third of each of these intervals resulting in $I_2 = [0, 1/9] \cup [2/9] \cup [2/3, 5/9] \cup [8/9, 1]$ and continue doing this forever. Show the intersection of all these I_n is nonempty. Letting $P = \cap_{n=1}^\infty I_n$ explain why every point of P is a limit point of P. Would the conclusion be any different if, instead of the middle third open interval, you took out an open interval of arbitrary length, each time leaving two closed intervals where there was one to begin with? This process produces something called the Cantor set. It is the basis for many pathological examples of unbelievably sick functions as well as being an essential ingredient in some extremely important theorems.

16. In Problem 15 in the case where the middle third is taken out, show the total length of open intervals removed equals 1. Thus what is left is very "short". For your information, the Cantor set is uncountable. In addition, it can be shown there exists a function which maps the Cantor set onto $[0,1]$, for example, although you could replace $[0,1]$ with the square $[0,1] \times [0,1]$ or more generally, any compact metric space, something you may study later.

17. Suppose $\{H_n\}$ is a sequence of sets with the property that for every point, x, there exists $r > 0$ such that $B(x, r)$ intersects only finitely many of the H_n. Such a collection of sets is called locally finite. Show that if the sets are all closed in addition to being locally finite, then the union of all these sets is also closed. This concept of local finiteness is of great significance although it will not be pursued further here.

18. A set, K is called compact if whenever $K \subseteq \cup\mathcal{C}$ for $\mathcal{C}$ a set whose elements are open sets, then there are finitely many of the open sets in $\mathcal{C}$, $U_1, \cdots, U_m$ such that
$$K \subseteq \cup_{k=1}^m U_k.$$
Show every closed interval, $[a, b]$ is compact. Next show every closed subset of $[a, b]$ is compact. **Hint:** For the first part, use the nested interval theorem in a manner similar to what was done to show $[a, b]$ is sequentially compact.

19. Show every closed and bounded subset of $\mathbb{F}$ is compact. **Hint:** You might first show every set of the form $[a, b] + i [c, d]$ is compact by considering sequences of nested intervals in both $[a, b]$ and $[c, d]$ where the nested intervals are obtained as in Problem 18.

20. Show a set, K is compact if and only if whenever $K \subseteq \cup\mathcal{B}$ where $\mathcal{B}$ is a set whose elements are open balls, it follows there are finitely many of these sets, $B_1, \cdots, B_m$ such that
$$K \subseteq \cup_{k=1}^m B_k$$
In words, every open cover of open balls admits a finite subcover.

21. Show every sequentially compact set in $\mathbb{F}$ is a closed subset of some disc $D(0, r)$ where this notation is explained in Proposition 4.7.12. From Problem 19, what does this say about sequentially compact sets being compact? Explain.

22. Now suppose K is a compact subset of $\mathbb{F}$ as explained in Problem 18. Show that K must be contained in some set of the form $[-r, r] + i [-r, r]$. When you have done this, show K must be sequentially compact. **Hint:** If the first part were not so, $\{(-n, n)\}_{n=1}^\infty$ would be an open cover but, does it have a finite subcover? For the second part, you know $K \subseteq [-r, r] + i [-r, r]$ for some r. Now if $\{x_n + iy_n\}_{n=1}^\infty$ is a sequence which has no subsequence which converges to a point in K, you know from Proposition 4.7.12 and Theorem 19, since $[-r, r] + i [-r, r]$ is sequentially compact, there is a subsequence, $\{x_{n_k} + iy_{n_k}\}_{k=1}^\infty$ which converges to some $x + iy \in [-r, r] + i [-r, r]$. Suppose $x + iy \notin K$ and consider the open cover of K given by $\{O_n\}_{n=1}^\infty$ where
$$O_n \equiv \{y : |y - x| > 1/n\}.$$
You need to verify the O_n are open sets and that they are an open cover of K which admits no finite subcover. From this you get a contradiction.

23. Show that every uncountable set of points in $\mathbb{F}$ has a limit point. This is not necessarily true if you replace the word, uncountable with the word, infinite. Explain why.

4.9 Cauchy Sequences And Completeness

You recall the definition of completeness which stated that every nonempty set of real numbers which is bounded above has a least upper bound and that every nonempty set of real numbers which is bounded below has a greatest lower bound and this is a property of the real line known as the completeness axiom. Geometrically, this involved filling in the holes. There is another way of describing completeness in terms of Cauchy sequences which will be discussed soon.

Definition 4.9.1 *$\{a_n\}$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists n_ε such that whenever $n, m \geq n_\varepsilon$,*

$$|a_n - a_m| < \varepsilon.$$

A sequence is Cauchy means the terms are "bunching up to each other" as m, n get large.

Theorem 4.9.2 *The set of terms (values) of a Cauchy sequence in $\mathbb{F}$ is bounded.*

Proof: Let $\varepsilon = 1$ in the definition of a Cauchy sequence and let $n > n_1$. Then from the definition,

$$|a_n - a_{n_1}| < 1.$$

It follows that for all $n > n_1$,

$$|a_n| < 1 + |a_{n_1}|.$$

Therefore, for all n,

$$|a_n| \leq 1 + |a_{n_1}| + \sum_{k=1}^{n_1} |a_k|.$$

This proves the theorem.

Theorem 4.9.3 *If a sequence $\{a_n\}$ in $\mathbb{F}$ converges, then the sequence is a Cauchy sequence.*

Proof: Let $\varepsilon > 0$ be given and suppose $a_n \to a$. Then from the definition of convergence, there exists n_ε such that if $n > n_\varepsilon$, it follows that

$$|a_n - a| < \frac{\varepsilon}{2}$$

Therefore, if $m, n \geq n_\varepsilon + 1$, it follows that

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

showing that, since $\varepsilon > 0$ is arbitrary, $\{a_n\}$ is a Cauchy sequence.

The following theorem is very useful.

Theorem 4.9.4 *Suppose $\{a_n\}$ is a Cauchy sequence in $\mathbb{F}$ and there exists a subsequence, $\{a_{n_k}\}$ which converges to a. Then $\{a_n\}$ also converges to a.*

Proof: Let $\varepsilon > 0$ be given. There exists N such that if $m, n > N$, then

$$|a_m - a_n| < \varepsilon/2.$$

Also there exists K such that if $k > K$, then

$$|a - a_{n_k}| < \varepsilon/2.$$

Then let $k > \max(K, N)$. Then for such k,

$$\begin{aligned} |a_k - a| &\leq |a_k - a_{n_k}| + |a_{n_k} - a| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves the theorem.

The next definition has to do with sequences which are real numbers.

Definition 4.9.5 *The sequence of real numbers,* $\{a_n\}$ *, is monotone increasing if for all* n*,* $a_n \leq a_{n+1}$*. The sequence is monotone decreasing if for all* n*,* $a_n \geq a_{n+1}$*. People often leave off the word "monotone".*

If someone says a sequence is monotone, it usually means monotone increasing.

There exist different descriptions of completeness. An important result is the following theorem which gives a version of completeness in terms of Cauchy sequences. This is often more convenient to use than the earlier definition in terms of least upper bounds and greatest lower bounds because this version of completeness, although it is equivalent to the completeness axiom for the real line, also makes sense in many situations where Definition 2.9.1 on Page 28 does not make sense, $\mathbb{C}$ for example because by Problem 12 on Page 33 there is no way to place an order on $\mathbb{C}$. This is also the case whenever the sequence is of points in multiple dimensions.

It is the concept of completeness and the notion of limits which sets analysis apart from algebra. You will find that every existence theorem in analysis depends on the assumption that some space is complete.

Theorem 4.9.6 *Every Cauchy sequence in* $\mathbb{R}$ *converges if and only if every nonempty set of real numbers which is bounded above has a least upper bound and every nonempty set of real numbers which is bounded below has a greatest lower bound.*

Proof: First suppose every Cauchy sequence converges and let S be a nonempty set which is bounded above. Let b_1 be an upper bound. Pick $s_1 \in S$. If $s_1 = b_1$, the upper least upper bound has been found and equals b_1. If $(s_1 + b_1)/2$ is an upper bound to S, let this equal b_2. If not, there exists $b_1 > s_2 > (s_1 + b_1)/2$ so let $b_2 = b_1$ and s_2 be as just described. Now let b_2 and s_2 play the same role as s_1 and b_1 and do the same argument. This yields a sequence $\{s_n\}$ of points of S which is monotone increasing and another sequence of upper bounds, $\{b_n\}$ which is monotone decreasing and

$$|s_n - b_n| \leq 2^{-n+1}(b_1 - s_1)$$

Therefore, if $m > n$

$$|b_n - b_m| \leq b_n - s_m \leq b_n - s_n \leq 2^{-n+1}(b_1 - s_1)$$

and so $\{b_n\}$ is a Cauchy sequence. Therefore, it converges to some number b. Then b must be an upper bound of S because if not, there would exist $s > b$ and then

$$b_n - b \geq s - b$$

which would prevent $\{b_n\}$ from converging to b. The claim that every nonempty set of numbers bounded below has a greatest lower bound follows similarly. Alternatively, you could consider $-S \equiv \{-x : x \in S\}$ and apply what was just shown.

Now suppose the condition about existence of least upper bounds and greatest lower bounds. Let $\{a_n\}$ be a Cauchy sequence. Then by Theorem 4.9.2 $\{a_n\} \subseteq [a, b]$ for some numbers a, b. By Theorem 4.7.2 there is a subsequence, $\{a_{n_k}\}$ which converges to $x \in [a, b]$. By Theorem 4.9.4, the original sequence converges to x also. This proves the theorem.

Theorem 4.9.7 *If either of the above conditions for completeness holds, then whenever* $\{a_n\}$ *is a monotone increasing sequence which is bounded above, it converges and whenever* $\{b_n\}$ *is a monotone sequence which is bounded below, it converges.*

Proof: Let $a = \sup\{a_n : n \geq 1\}$ and let $\varepsilon > 0$ be given. Then from Proposition 2.9.3 on Page 28 there exists m such that $a - \varepsilon < a_m \leq a$. Since the sequence is increasing, it follows that for all $n \geq m$, $a - \varepsilon < a_n \leq a$. Thus $a = \lim_{n\to\infty} a_n$. The case of a decreasing sequence is similar. Alternatively, you could consider the sequence $\{-a_n\}$ and apply what was just shown to this decreasing sequence. This proves the theorem.

By Theorem 4.9.6 the following definition of completeness is equivalent to the original definition when both apply.

Definition 4.9.8 *Whenever every Cauchy sequence in some set converges, the set is called complete.*

Theorem 4.9.9 $\mathbb{F}$ *is complete.*

Proof: Suppose $\{x_n + iy_n\}_{n=1}^{\infty}$ be a Cauchy sequence. This happens if and only if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences because

$$\begin{aligned} |x_m - x_n| &\leq |x_m + iy_m - (x_n + iy_n)| \\ |y_m - y_n| &\leq |x_m + iy_m - (x_n + iy_n)| \end{aligned}$$

and the right hand side is assumed to be small whenever m, n are large enough. Thus there exists x such that $x_n \to x$ and y such that $y \to y$. Hence, $x_n + iy_n \to x + iy$.

4.9.1 Decimals

You are all familiar with decimals. In the United States these are written in the form $.a_1a_2a_3\cdots$ where the a_i are integers between 0 and 9.[2] Thus .23417432 is a number written as a decimal. You also recall the meaning of such notation in the case of a terminating decimal. For example, .234 is defined as $\frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3}$. Now what is meant by a nonterminating decimal?

Definition 4.9.10 *Let* $.a_1a_2\cdots$ *be a decimal. Define*

$$.a_1a_2\cdots \equiv \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k}.$$

Proposition 4.9.11 *The above definition makes sense.*

Proof: Note the sequence $\left\{\sum_{k=1}^{n} \frac{a_k}{10^k}\right\}_{n=1}^{\infty}$ is an increasing sequence. Therefore, if there exists an upper bound, it follows from Theorem 4.9.7 that this sequence converges and so the definition is well defined.

$$\sum_{k=1}^{n} \frac{a_k}{10^k} \leq \sum_{k=1}^{n} \frac{9}{10^k} = 9\sum_{k=1}^{n} \frac{1}{10^k}.$$

Now

$$\begin{aligned} \frac{9}{10}\left(\sum_{k=1}^{n} \frac{1}{10^k}\right) &= \sum_{k=1}^{n} \frac{1}{10^k} - \frac{1}{10}\sum_{k=1}^{n} \frac{1}{10^k} = \sum_{k=1}^{n} \frac{1}{10^k} - \sum_{k=2}^{n+1} \frac{1}{10^k} \\ &= \frac{1}{10} - \frac{1}{10^{n+1}} \end{aligned}$$

[2] In France and Russia they use a comma instead of a period. This looks very strange but that is just the way they do it.

and so

$$\sum_{k=1}^{n} \frac{1}{10^k} \leq \frac{10}{9}\left(\frac{1}{10} - \frac{1}{10^{n+1}}\right) \leq \frac{10}{9}\left(\frac{1}{10}\right) = \frac{1}{9}.$$

Therefore, since this holds for all n, it follows the above sequence is bounded above. It follows the limit exists.

4.9.2 $\limsup$ and $\liminf$

Sometimes the limit of a sequence does not exist. For example, if $a_n = (-1)^n$, then $\lim_{n\to\infty} a_n$ does not exist. This is because the terms of the sequence are a distance of 1 apart. Therefore there can't exist a single number such that all the terms of the sequence are ultimately within 1/4 of that number. The nice thing about $\limsup$ and $\liminf$ is that they always exist. First here is a simple lemma and definition. First review the definition of inf and sup on Page 28 along with the simple properties of these things.

Definition 4.9.12 *Denote by $[-\infty, \infty]$ the real line along with symbols ∞ and $-\infty$. It is understood that ∞ is larger than every real number and $-\infty$ is smaller than every real number. Then if $\{A_n\}$ is an increasing sequence of points of $[-\infty, \infty]$, $\lim_{n\to\infty} A_n$ equals ∞ if the only upper bound of the set $\{A_n\}$ is ∞. If $\{A_n\}$ is bounded above by a real number, then $\lim_{n\to\infty} A_n$ is defined in the usual way and equals the least upper bound of $\{A_n\}$. If $\{A_n\}$ is a decreasing sequence of points of $[-\infty, \infty]$, $\lim_{n\to\infty} A_n$ equals $-\infty$ if the only lower bound of the sequence $\{A_n\}$ is $-\infty$. If $\{A_n\}$ is bounded below by a real number, then $\lim_{n\to\infty} A_n$ is defined in the usual way and equals the greatest lower bound of $\{A_n\}$. More simply, if $\{A_n\}$ is increasing,*

$$\lim_{n\to\infty} A_n \equiv \sup\{A_n\}$$

and if $\{A_n\}$ is decreasing then

$$\lim_{n\to\infty} A_n \equiv \inf\{A_n\}.$$

Lemma 4.9.13 *Let $\{a_n\}$ be a sequence of real numbers and let $U_n \equiv \sup\{a_k : k \geq n\}$. Then $\{U_n\}$ is a decreasing sequence. Also if $L_n \equiv \inf\{a_k : k \geq n\}$, then $\{L_n\}$ is an increasing sequence. Therefore, $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} U_n$ both exist.*

Proof: Let W_n be an upper bound for $\{a_k : k \geq n\}$. Then since these sets are getting smaller, it follows that for $m < n$, W_m is an upper bound for $\{a_k : k \geq n\}$. In particular if $W_m = U_m$, then U_m is an upper bound for $\{a_k : k \geq n\}$ and so U_m is at least as large as U_n, the least upper bound for $\{a_k : k \geq n\}$. The claim that $\{L_n\}$ is decreasing is similar. This proves the lemma.

From the lemma, the following definition makes sense.

Definition 4.9.14 *Let $\{a_n\}$ be any sequence of points of $[-\infty, \infty]$*

$$\limsup_{n\to\infty} a_n \equiv \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

$$\liminf_{n\to\infty} a_n \equiv \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

Theorem 4.9.15 *Suppose $\{a_n\}$ is a sequence of real numbers and that $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ and in this case,*

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof: First note that

$$\sup\{a_k : k \geq n\} \geq \inf\{a_k : k \geq n\}$$

and so from Theorem 4.4.11,

$$\begin{aligned}\limsup_{n\to\infty} a_n &\equiv \lim_{n\to\infty} \sup\{a_k : k \geq n\} \\ &\geq \lim_{n\to\infty} \inf\{a_k : k \geq n\} \\ &\equiv \liminf_{n\to\infty} a_n.\end{aligned}$$

Suppose first that $\lim_{n\to\infty} a_n$ exists and is a real number. Then by Theorem 4.9.3 $\{a_n\}$ is a Cauchy sequence. Therefore, if $\varepsilon > 0$ is given, there exists N such that if $m, n \geq N$, then

$$|a_n - a_m| < \varepsilon/3.$$

From the definition of $\sup\{a_k : k \geq N\}$, there exists $n_1 \geq N$ such that

$$\sup\{a_k : k \geq N\} \leq a_{n_1} + \varepsilon/3.$$

Similarly, there exists $n_2 \geq N$ such that

$$\inf\{a_k : k \geq N\} \geq a_{n_2} - \varepsilon/3.$$

It follows that

$$\sup\{a_k : k \geq N\} - \inf\{a_k : k \geq N\} \leq |a_{n_1} - a_{n_2}| + \frac{2\varepsilon}{3} < \varepsilon.$$

Since the sequence, $\{\sup\{a_k : k \geq N\}\}_{N=1}^{\infty}$ is decreasing and $\{\inf\{a_k : k \geq N\}\}_{N=1}^{\infty}$ is increasing, it follows from Theorem 4.4.11

$$0 \leq \lim_{N\to\infty} \sup\{a_k : k \geq N\} - \lim_{N\to\infty} \inf\{a_k : k \geq N\} \leq \varepsilon$$

Since ε is arbitrary, this shows

$$\lim_{N\to\infty} \sup\{a_k : k \geq N\} = \lim_{N\to\infty} \inf\{a_k : k \geq N\} \tag{4.8}$$

Next suppose 4.8. Then

$$\lim_{N\to\infty} (\sup\{a_k : k \geq N\} - \inf\{a_k : k \geq N\}) = 0$$

Since $\sup\{a_k : k \geq N\} \geq \inf\{a_k : k \geq N\}$ it follows that for every $\varepsilon > 0$, there exists N such that

$$\sup\{a_k : k \geq N\} - \inf\{a_k : k \geq N\} < \varepsilon$$

Thus if $m, n > N$, then

$$|a_m - a_n| < \varepsilon$$

which means $\{a_n\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, it follows from Theorem 4.9.6 that $\lim_{n\to\infty} a_n \equiv a$ exists. By Theorem 4.4.7, the squeezing theorem, it follows

$$a = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and this proves the theorem.

With the above theorem, here is how to define the limit of a sequence of points in $[-\infty, \infty]$.

Definition 4.9.16 *Let* $\{a_n\}$ *be a sequence of points of* $[-\infty,\infty]$. *Then* $\lim_{n\to\infty} a_n$ *exists exactly when*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and in this case

$$\lim_{n\to\infty} a_n \equiv \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

The significance of lim sup and lim inf, in addition to what was just discussed, is contained in the following theorem which follows quickly from the definition.

Theorem 4.9.17 *Suppose* $\{a_n\}$ *is a sequence of points of* $[-\infty,\infty]$. *Let*

$$\lambda = \limsup_{n\to\infty} a_n.$$

Then if $b > \lambda$, *it follows there exists* N *such that whenever* $n \geq N$,

$$a_n \leq b.$$

If $c < \lambda$, *then* $a_n > c$ *for infinitely many values of* n. *Let*

$$\gamma = \liminf_{n\to\infty} a_n.$$

Then if $d < \gamma$, *it follows there exists* N *such that whenever* $n \geq N$,

$$a_n \geq d.$$

If $e > \gamma$, *it follows* $a_n < e$ *for infinitely many values of* n.

The proof of this theorem is left as an exercise for you. It follows directly from the definition and it is the sort of thing you must do yourself. Here is one other simple proposition.

Proposition 4.9.18 *Let* $\lim_{n\to\infty} a_n = a > 0$. *Then*

$$\limsup_{n\to\infty} a_n b_n = a \limsup_{n\to\infty} b_n.$$

Proof: This follows from the definition. Let $\lambda_n = \sup\{a_k b_k : k \geq n\}$. For all n large enough, $a_n > a - \varepsilon$ where ε is small enough that $a - \varepsilon > 0$. Therefore,

$$\lambda_n \geq \sup\{b_k : k \geq n\}(a - \varepsilon)$$

for all n large enough. Then

$$\begin{aligned}
\limsup_{n\to\infty} a_n b_n &= \lim_{n\to\infty} \lambda_n \equiv \limsup_{n\to\infty} a_n b_n \\
&\geq \lim_{n\to\infty} \left(\sup\{b_k : k \geq n\}(a - \varepsilon)\right) \\
&= (a - \varepsilon) \limsup_{n\to\infty} b_n
\end{aligned}$$

Similar reasoning shows

$$\limsup_{n\to\infty} a_n b_n \leq (a + \varepsilon) \limsup_{n\to\infty} b_n$$

Now since $\varepsilon > 0$ is arbitrary, the conclusion follows.

4.9.3 Shrinking Diameters

It is useful to consider another version of the nested interval lemma. This involves a sequence of sets such that set $(n+1)$ is contained in set n and such that their diameters converge to 0. It turns out that if the sets are also closed, then often there exists a unique point in all of them.

Definition 4.9.19 *Let S be a nonempty set. Then* $\text{diam}(S)$ *is defined as*

$$\text{diam}(S) \equiv \sup\{|x-y| : x, y \in S\}.$$

This is called the diameter of S.

Theorem 4.9.20 *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of closed sets in $\mathbb{F}$ such that*

$$\lim_{n\to\infty} \text{diam}(F_n) = 0$$

and $F_n \supseteq F_{n+1}$ for each n. Then there exists a unique $p \in \cap_{k=1}^{\infty} F_k$.

Proof: Pick $p_k \in F_k$. This is always possible because by assumption each set is nonempty. Then $\{p_k\}_{k=m}^{\infty} \subseteq F_m$ and since the diameters converge to 0 it follows $\{p_k\}$ is a Cauchy sequence. Therefore, it converges to a point, p by completeness of $\mathbb{F}$. Since each F_k is closed, it must be that $p \in F_k$ for all k. Therefore, $p \in \cap_{k=1}^{\infty} F_k$. If $q \in \cap_{k=1}^{\infty} F_k$, then since both $p, q \in F_k$,

$$|p-q| \leq \text{diam}(F_k).$$

It follows since these diameters converge to 0, $|p-q| \leq \varepsilon$ for every ε. Hence $p = q$. This proves the theorem.

A sequence of sets, $\{G_n\}$ which satisfies $G_n \supseteq G_{n+1}$ for all n is called a nested sequence of sets.

4.10 Exercises

1. Suppose $x = .3434343\overline{34}$ where the bar over the last 34 signifies that this repeats forever. In elementary school you were probably given the following procedure for finding the number x as a quotient of integers. First multiply by 100 to get $100x = 34.343434\overline{34}$ and then subtract to get $99x = 34$. From this you conclude that $x = 34/99$. Fully justify this procedure. **Hint:** $.343434\overline{34} = \lim_{n\to\infty} 34 \sum_{k=1}^{n} \left(\frac{1}{100}\right)^k$ now use Problem 7.

2. Let $a \in [0,1]$. Show $a = .a_1a_2a_3\cdots$ for some choice of integers, $a_1, a_2, \cdots$ if it is possible to do this. Give an example where there may be more than one way to do this.

3. Show every rational number between 0 and 1 has a decimal expansion which either repeats or terminates.

4. Consider the number whose decimal expansion is $.010010001000010000010000001\cdots$. Show this is an irrational number.

5. Show that between any two integers there exists an irrational number. Next show that between any two numbers there exists an irrational number.

6. Let a be a positive number and let $x_1 = b > 0$ where $b^2 > a$. Explain why there exists such a number, b. Now having defined x_n, define $x_{n+1} \equiv \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. Verify that $\{x_n\}$ is a decreasing sequence and that it satisfies $x_n^2 \geq a$ for all n and is therefore, bounded below. Explain why $\lim_{n\to\infty} x_n$ exists. If x is this limit, show that $x^2 = a$. Explain how this shows that every positive real number has a square root. This is an example of a recursively defined sequence. Note this does not give a formula for x_n, just a rule which tells how to define x_{n+1} if x_n is known.

7. Let $a_1 = 0$ and suppose that $a_{n+1} = \frac{9}{9-a_n}$. Write a_2, a_3, a_4. Now prove that for all n, it follows that $a_n \leq \frac{9}{2} + \frac{3}{2}\sqrt{5}$ (By Problem 6 there exists $\sqrt{5}$.) and so the sequence is bounded above. Next show that the sequence is increasing and so it converges. Find the limit of the sequence. **Hint:** You should prove these things by induction. Finally, to find the limit, let $n \to \infty$ in both sides and argue that the limit, a, must satisfy $a = \frac{9}{9-a}$.

8. Prove $\sqrt{2}$ is irrational. **Hint:** Suppose $\sqrt{2} = p/q$ where p, q are positive integers and the fraction is in lowest terms. Then $2q^2 = p^2$ and so p^2 is even. Explain why $p = 2r$ so p must be even. Next argue q must be even. Use this to show that between any two real numbers there is an irrational number. That is, the irrational numbers are dense.

9. If $\lim_{n\to\infty} a_n = a$, does it follow that $\lim_{n\to\infty} |a_n| = |a|$? Prove or else give a counter example.

10. Show the following converge to 0.

    (a) $\frac{n^5}{1.01^n}$

    (b) $\frac{10^n}{n!}$

11. Suppose $\lim_{n\to\infty} x_n = x$. Show that then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = x$. Give an example where $\lim_{n\to\infty} x_n$ does not exist but $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k$ does.

12. Suppose $r \in (0,1)$. Show that $\lim_{n\to\infty} r^n = 0$. **Hint:** Use the binomial theorem. $r = \frac{1}{1+\delta}$ where $\delta > 0$. Therefore, $r^n = \frac{1}{(1+\delta)^n} < \frac{1}{1+n\delta}$, etc.

13. Let $p \in \mathbb{N}$ and $\alpha > 0$ and let $x_1 > 0$ and
$$x_{n+1} \equiv \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{1-p}.$$
Show $\lim_{n\to\infty} x_n = x$ where x is a positive number such that $x^p = \alpha$. This shows the existence of a p^{th} root for any positive number. **Hint:** It will help if you first use the binomial theorem to show that for all $x, y > 0$
$$y^p \geq x^p + px^{p-1}(y - x).$$
This little inequality can be used to show each x_n satisfies $x_n^p \geq \alpha$. Put x_n for x and y replaced with x_{n+1}. Then it is easy to show the sequence is decreasing.

14. Prove $\lim_{n\to\infty} \sqrt[n]{n} = 1$. **Hint:** Let $e_n \equiv \sqrt[n]{n} - 1$ so that $(1 + e_n)^n = n$. Now observe that $e_n > 0$ and use the binomial theorem to conclude $1 + ne_n + \frac{n(n-1)}{2}e_n^2 \leq n$. This nice approach to establishing this limit using only elementary algebra is in Rudin [30].

15. Find $\lim_{n\to\infty} (x^n + 5)^{1/n}$ for $x \geq 0$. There are two cases here, $x \leq 1$ and $x > 1$. Show that if $x > 1$, the limit is x while if $x \leq 1$ the limit equals 1. **Hint:** Use the argument of Problem 14. This interesting example is in [9].

16. Find $\limsup_{n\to\infty} (-1)^n$ and $\liminf_{n\to\infty} (-1)^n$. Explain your conclusions.

17. Give a careful proof of Theorem 4.9.17.

18. Let $\lambda = \limsup_{n\to\infty} a_n$. Show there exists a subsequence, $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = \lambda$. Now consider the set, S of all points in $[-\infty, \infty]$ such that for $s \in S$, some subsequence of $\{a_n\}$ converges to s. Show that S has a largest point and this point is $\limsup_{n\to\infty} a_n$.

19. Let $\lambda = \liminf_{n\to\infty} a_n$. Show there exists a subsequence, $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = \lambda$. Now consider the set, S of all points in $[-\infty, \infty]$ such that for $s \in S$, some subsequence of $\{a_n\}$ converges to s. Show that S has a smallest point and this point is $\liminf_{n\to\infty} a_n$.

20. Prove that if $a_n \leq b_n$ for all n sufficiently large that
$$\begin{aligned}\liminf_{n\to\infty} a_n &\leq \liminf_{n\to\infty} b_n,\\ \limsup_{n\to\infty} a_n &\leq \limsup_{n\to\infty} b_n.\end{aligned}$$

21. Prove that
$$\limsup_{n\to\infty} (-a_n) = -\liminf_{n\to\infty} a_n.$$

22. Prove that if $a \geq 0$, then
$$\limsup_{n\to\infty} aa_n = a \limsup_{n\to\infty} a_n$$
while if $a < 0$,
$$\limsup_{n\to\infty} aa_n = a \liminf_{n\to\infty} a_n.$$

23. Prove that if $\lim_{n\to\infty} b_n = b$, then
$$\limsup_{n\to\infty} (b_n + a_n) = b + \limsup_{n\to\infty} a_n.$$
Conjecture and prove a similar result for $\liminf$.

24. Give conditions under which the following inequalities hold.
$$\begin{aligned}\limsup_{n\to\infty} (a_n + b_n) &\leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n\\ \liminf_{n\to\infty} (a_n + b_n) &\geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.\end{aligned}$$
Hint: You need to consider whether the right hand sides make sense. Thus you can't consider $-\infty + \infty$.

25. Give an example of a nested sequence of nonempty sets whose diameters converge to 0 which have no point in their intersection.

26. Give an example of a nested sequence of nonempty sets whose intersection has more than one point. Next give an example of a nested sequence of nonempty sets which has 2 points in their intersection.

Infinite Series Of Numbers

5.1 Basic Considerations

Earlier in Definition 4.4.1 on Page 49 the notion of limit of a sequence was discussed. There is a very closely related concept called an infinite series which is dealt with in this section.

Definition 5.1.1 *Define*

$$\sum_{k=m}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=m}^{n} a_k$$

whenever the limit exists and is finite. In this case the series is said to converge. If it does not converge, it is said to diverge. The sequence $\left\{\sum_{k=m}^{n} a_k\right\}_{n=m}^{\infty}$ *in the above is called the sequence of partial sums.*

From this definition, it should be clear that infinite sums do not always make sense. Sometimes they do and sometimes they don't, depending on the behavior of the partial sums. As an example, consider $\sum_{k=1}^{\infty} (-1)^k$. The partial sums corresponding to this symbol alternate between -1 and 0. Therefore, there is no limit for the sequence of partial sums. It follows the symbol just written is meaningless and the infinite sum diverges.

Example 5.1.2 *Find the infinite sum,* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Note $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and so $\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) = -\frac{1}{N+1} + 1$. Therefore,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = \lim_{N\to\infty} \left(-\frac{1}{N+1} + 1\right) = 1.$$

Proposition 5.1.3 *Let* $a_k \geq 0$. *Then* $\left\{\sum_{k=m}^{n} a_k\right\}_{n=m}^{\infty}$ *is an increasing sequence. If this sequence is bounded above, then* $\sum_{k=m}^{\infty} a_k$ *converges and its value equals*

$$\sup\left\{\sum_{k=m}^{n} a_k : n = m, m+1, \cdots\right\}.$$

When the sequence is not bounded above, $\sum_{k=m}^{\infty} a_k$ *diverges.*

Proof: It follows $\left\{\sum_{k=m}^{n} a_k\right\}_{n=m}^{\infty}$ is an increasing sequence because

$$\sum_{k=m}^{n+1} a_k - \sum_{k=m}^{n} a_k = a_{n+1} \geq 0.$$

If it is bounded above, then by the form of completeness found in Theorem 4.9.6 on Page 61 it follows the sequence of partial sums converges to $\sup\{\sum_{k=m}^{n} a_k : n = m, m+1, \cdots\}$. If the sequence of partial sums is not bounded, then it is not a Cauchy sequence and so it does not converge. See Theorem 4.9.3 on Page 60. This proves the proposition.

In the case where $a_k \geq 0$, the above proposition shows there are only two alternatives available. Either the sequence of partial sums is bounded above or it is not bounded above. In the first case convergence occurs and in the second case, the infinite series diverges. For this reason, people will sometimes write $\sum_{k=m}^{\infty} a_k < \infty$ to denote the case where convergence occurs and $\sum_{k=m}^{\infty} a_k = \infty$ for the case where divergence occurs. Be very careful you never think this way in the case where it is not true that all $a_k \geq 0$. For example, the partial sums of $\sum_{k=1}^{\infty} (-1)^k$ are bounded because they are all either -1 or 0 but the series does not converge.

One of the most important examples of a convergent series is the geometric series. This series is $\sum_{n=0}^{\infty} r^n$. The study of this series depends on simple high school algebra and Theorem 4.4.9 on Page 51. Let $S_n \equiv \sum_{k=0}^{n} r^k$. Then

$$S_n = \sum_{k=0}^{n} r^k, \; rS_n = \sum_{k=0}^{n} r^{k+1} = \sum_{k=1}^{n+1} r^k.$$

Therefore, subtracting the second equation from the first yields

$$(1-r) S_n = 1 - r^{n+1}$$

and so a formula for S_n is available. In fact, if $r \neq 1$,

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

By Theorem 4.4.9, $\lim_{n\to\infty} S_n = \frac{1}{1-r}$ in the case when $|r| < 1$. Now if $|r| \geq 1$, the limit clearly does not exist because S_n fails to be a Cauchy sequence (Why?). This shows the following.

Theorem 5.1.4 *The geometric series, $\sum_{n=0}^{\infty} r^n$ converges and equals $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.*

If the series do converge, the following holds.

Theorem 5.1.5 *If $\sum_{k=m}^{\infty} a_k$ and $\sum_{k=m}^{\infty} b_k$ both converge and x, y are numbers, then*

$$\sum_{k=m}^{\infty} a_k = \sum_{k=m+j}^{\infty} a_{k-j} \tag{5.1}$$

$$\sum_{k=m}^{\infty} x a_k + y b_k = x \sum_{k=m}^{\infty} a_k + y \sum_{k=m}^{\infty} b_k \tag{5.2}$$

$$\left| \sum_{k=m}^{\infty} a_k \right| \leq \sum_{k=m}^{\infty} |a_k| \tag{5.3}$$

where in the last inequality, the last sum equals $+\infty$ if the partial sums are not bounded above.

Proof: The above theorem is really only a restatement of Theorem 4.4.6 on Page 50 and the above definitions of infinite series. Thus

$$\sum_{k=m}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=m}^{n} a_k = \lim_{n\to\infty} \sum_{k=m+j}^{n+j} a_{k-j} = \sum_{k=m+j}^{\infty} a_{k-j}.$$

To establish 5.2, use Theorem 4.4.6 on Page 50 to write

$$\begin{aligned} \sum_{k=m}^{\infty} xa_k + yb_k &= \lim_{n\to\infty} \sum_{k=m}^{n} xa_k + yb_k \\ &= \lim_{n\to\infty} \left(x \sum_{k=m}^{n} a_k + y \sum_{k=m}^{n} b_k \right) \\ &= x \sum_{k=m}^{\infty} a_k + y \sum_{k=m}^{\infty} b_k. \end{aligned}$$

Formula 5.3 follows from the observation that, from the triangle inequality,

$$\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{\infty} |a_k|$$

and so

$$\left| \sum_{k=m}^{\infty} a_k \right| = \lim_{n\to\infty} \left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{\infty} |a_k| .$$

Example 5.1.6 *Find* $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{6}{3^n} \right)$.

From the above theorem and Theorem 5.1.4,

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{6}{3^n} \right) &= 5 \sum_{n=0}^{\infty} \frac{1}{2^n} + 6 \sum_{n=0}^{\infty} \frac{1}{3^n} \\ &= 5 \frac{1}{1-(1/2)} + 6 \frac{1}{1-(1/3)} = 19. \end{aligned}$$

The following criterion is useful in checking convergence.

Theorem 5.1.7 *Let* $\{a_k\}$ *be a sequence of points in* $\mathbb{F}$. *The sum* $\sum_{k=m}^{\infty} a_k$ *converges if and only if for all* $\varepsilon > 0$, *there exists* n_ε *such that if* $q \geq p \geq n_\varepsilon$, *then*

$$\left| \sum_{k=p}^{q} a_k \right| < \varepsilon. \tag{5.4}$$

Proof: Suppose first that the series converges. Then $\{\sum_{k=m}^{n} a_k\}_{n=m}^{\infty}$ is a Cauchy sequence by Theorem 4.9.3 on Page 60. Therefore, there exists $n_\varepsilon > m$ such that if $q \geq p-1 \geq n_\varepsilon > m$,

$$\left| \sum_{k=m}^{q} a_k - \sum_{k=m}^{p-1} a_k \right| = \left| \sum_{k=p}^{q} a_k \right| < \varepsilon. \tag{5.5}$$

Next suppose 5.4 holds. Then from 5.5 it follows upon letting p be replaced with $p+1$ that $\{\sum_{k=m}^{n} a_k\}_{n=m}^{\infty}$ is a Cauchy sequence and so, by Theorem 4.9.9, it converges. By the definition of infinite series, this shows the infinite sum converges as claimed.

Definition 5.1.8 *A series*

$$\sum_{k=m}^{\infty} a_k$$

is said to converge absolutely if

$$\sum_{k=m}^{\infty} |a_k|$$

converges. If the series does converge but does not converge absolutely, then it is said to converge conditionally.

Theorem 5.1.9 *If $\sum_{k=m}^{\infty} a_k$ converges absolutely, then it converges.*

Proof: Let $\varepsilon > 0$ be given. Then by assumption and Theorem 5.1.7, there exists n_ε such that whenever $q \geq p \geq n_\varepsilon$,

$$\sum_{k=p}^{q} |a_k| < \varepsilon.$$

Therefore, from the triangle inequality,

$$\varepsilon > \sum_{k=p}^{q} |a_k| \geq \left| \sum_{k=p}^{q} a_k \right|.$$

By Theorem 5.1.7, $\sum_{k=m}^{\infty} a_k$ converges and this proves the theorem.

In fact, the above theorem is really another version of the completeness axiom. Thus its validity implies completeness. You might try to show this.

Theorem 5.1.10 *(comparison test) Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of non negative real numbers and suppose for all n sufficiently large, $a_n \leq b_n$. Then*

1. *If $\sum_{n=k}^{\infty} b_n$ converges, then $\sum_{n=m}^{\infty} a_n$ converges.*
2. *If $\sum_{n=k}^{\infty} a_n$ diverges, then $\sum_{n=m}^{\infty} b_n$ diverges.*

Proof: Consider the first claim. From the assumption there exists n^* such that $n^* > \max(k, m)$ and for all $n \geq n^*$ $b_n \geq a_n$. Then if $p \geq n^*$,

$$\begin{aligned} \sum_{n=m}^{p} a_n &\leq \sum_{n=m}^{n^*} a_n + \sum_{n=n^*+1}^{k} b_n \\ &\leq \sum_{n=m}^{n^*} a_n + \sum_{n=k}^{\infty} b_n. \end{aligned}$$

Thus the sequence,$\left\{\sum_{n=m}^{p} a_n\right\}_{p=m}^{\infty}$ is bounded above and increasing. Therefore, it converges by completeness. The second claim is left as an exercise.

Example 5.1.11 *Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.*

For $n > 1$,

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)}.$$

Now

$$\sum_{n=2}^{p} \frac{1}{n(n-1)} = \sum_{n=2}^{p} \left[\frac{1}{n-1} - \frac{1}{n}\right] = 1 - \frac{1}{p} \to 1$$

Therefore, letting $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n-1)}$

A convenient way to implement the comparison test is to use the limit comparison test. This is considered next.

Theorem 5.1.12 *Let $a_n, b_n > 0$ and suppose for all n large enough,*

$$0 < a < \frac{a_n}{b_n} \leq \frac{a_n}{b_n} < b < \infty.$$

Then $\sum a_n$ and $\sum b_n$ converge or diverge together.

Proof: Let n^* be such that $n \geq n^*$, then

$$\frac{a_n}{b_n} > a \text{ and } \frac{a_n}{b_n} < b$$

and so for all such n,

$$ab_n < a_n < bb_n$$

and so the conclusion follows from the comparison test.

The following corollary follows right away from the definition of the limit.

Corollary 5.1.13 *Let $a_n, b_n > 0$ and suppose*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lambda \in (0, \infty).$$

Then $\sum a_n$ and $\sum b_n$ converge or diverge together.

Example 5.1.14 *Determine the convergence of* $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n^4+2n+7}}$.

This series converges by the limit comparison test above. Compare with the series of Example 5.1.11.

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{\sqrt{n^4+2n+7}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n^4+2n+7}}{n^2} = \lim_{n\to\infty} \sqrt{1 + \frac{2}{n^3} + \frac{7}{n^4}} = 1.$$

Therefore, the series converges with the series of Example 5.1.11. How did I know what to compare with? I noticed that $\sqrt{n^4+2n+7}$ is essentially like $\sqrt{n^4} = n^2$ for large enough n. You see, the higher order term, n^4 dominates the other terms in $n^4 + 2n + 7$. Therefore, reasoning that $1/\sqrt{n^4+2n+7}$ is a lot like $1/n^2$ for large n, it was easy to see what to compare with. Of course this is not always easy and there is room for acquiring skill through practice.

To really exploit this limit comparison test, it is desirable to get lots of examples of series, some which converge and some which do not. The tool for obtaining these examples here will be the following wonderful theorem known as the Cauchy condensation test.

Theorem 5.1.15 *Let $a_n \geq 0$ and suppose the terms of the sequence $\{a_n\}$ are decreasing. Thus $a_n \geq a_{n+1}$ for all n. Then*

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} 2^n a_{2^n}$$

converge or diverge together.

Proof: This follows from the inequality of the following claim.
Claim:

$$\sum_{k=1}^{n} 2^k a_{2^{k-1}} \geq \sum_{k=1}^{2^n} a_k \geq \sum_{k=0}^{n} 2^{k-1} a_{2^k}.$$

Proof of the Claim: Note the claim is true for $n = 1$. Suppose the claim is true for n. Then, since $2^{n+1} - 2^n = 2^n$, and the terms, a_n, are decreasing,

$$\sum_{k=1}^{n+1} 2^k a_{2^{k-1}} = 2^{n+1} a_{2^n} + \sum_{k=1}^{n} 2^k a_{2^{k-1}} \geq 2^{n+1} a_{2^n} + \sum_{k=1}^{2^n} a_k$$

$$\geq \sum_{k=1}^{2^{n+1}} a_k \geq 2^n a_{2^{n+1}} + \sum_{k=1}^{2^n} a_k \geq 2^n a_{2^{n+1}} + \sum_{k=0}^{n} 2^{k-1} a_{2^k} = \sum_{k=0}^{n+1} 2^{k-1} a_{2^k}.$$

Example 5.1.16 *Determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^p}$ where p is a positive number. These are called the p series.*

Let $a_n = \frac{1}{n^p}$. Then $a_{2^n} = \left(\frac{1}{2^p}\right)^n$. From the Cauchy condensation test the two series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^p}\right)^n = \sum_{n=0}^{\infty} \left(2^{(1-p)}\right)^n$$

converge or diverge together. If $p > 1$, the last series above is a geometric series having common ratio less than 1 and so it converges. If $p \leq 1$, it is still a geometric series but in this case the common ratio is either 1 or greater than 1 so the series diverges. It follows that the p series converges if $p > 1$ and diverges if $p \leq 1$. In particular, $\sum_{n=1}^{\infty} n^{-1}$ diverges while $\sum_{n=1}^{\infty} n^{-2}$ converges.

Example 5.1.17 *Determine the convergence of $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n^2+100n}}$.*

Use the limit comparison test.

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{\sqrt{n^2+100n}}\right)} = 1$$

and so this series diverges with $\sum_{k=1}^{\infty} \frac{1}{k}$.

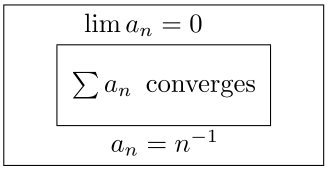
Sometimes it is good to be able to say a series does not converge. The n^{th} term test gives such a condition which is sufficient for this. It is really a corollary of Theorem 5.1.7.

Theorem 5.1.18 *If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof: Apply Theorem 5.1.7 to conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{k=n}^{n} a_k = 0.$$

It is very important to observe that this theorem goes only in one direction. That is, you cannot conclude the series converges if $\lim_{n\to\infty} a_n = 0$. If this happens, you don't know anything from this information. Recall $\lim_{n\to\infty} n^{-1} = 0$ but $\sum_{n=1}^{\infty} n^{-1}$ diverges. The following picture is descriptive of the situation.



5.2 Exercises

1. Determine whether the following series converge and give reasons for your answers.
   (a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n+1}}$
   (b) $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$
   (c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$
   (d) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$
   (e) $\sum_{n=1}^{\infty} \frac{1}{2n+2}$
   (f) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$
   (g) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$
2. Determine whether the following series converge and give reasons for your answers.
   (a) $\sum_{n=1}^{\infty} \frac{2^n+n}{n2^n}$
   (b) $\sum_{n=1}^{\infty} \frac{2^n+n}{n^2 2^n}$
   (c) $\sum_{n=1}^{\infty} \frac{n}{2n+1}$
   (d) $\sum_{n=1}^{\infty} \frac{n^{100}}{1.01^n}$
3. Find the exact values of the following infinite series if they converge.
   (a) $\sum_{k=3}^{\infty} \frac{1}{k(k-2)}$
   (b) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$
   (c) $\sum_{k=3}^{\infty} \frac{1}{(k+1)(k-2)}$
   (d) $\sum_{k=1}^{N} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$

4. Suppose $\sum_{k=1}^{\infty} a_k$ converges and each $a_k \geq 0$. Does it follow that $\sum_{k=1}^{\infty} a_k^2$ also converges?

5. Find a series which diverges using one test but converges using another if possible. If this is not possible, tell why.

6. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge and a_n, b_n are nonnegative, can you conclude the sum, $\sum_{n=1}^{\infty} a_n b_n$ converges?

7. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all n and b_n is bounded, can you conclude $\sum_{n=1}^{\infty} a_n b_n$ converges?

8. Determine the convergence of the series $\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k}\right)^{-n/2}$.

9. Is it possible there could exist a decreasing sequence of positive numbers, $\{a_n\}$ such that $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} \left(1 - \frac{a_{n+1}}{a_n}\right)$ converges? (This seems to be a fairly difficult problem.)

10. Suppose $\sum a_n$ converges conditionally and each a_n is real. Show it is possible to add the series in some order such that the result converges to 13. Then show it is possible to add the series in another order so that the result converges to 7. Thus there is no generalization of the commutative law for conditionally convergent infinite series.

11. If $\sum a_n$ converges absolutely, show that adding the terms in any order always gives the same answer.

5.3 More Tests For Convergence

5.3.1 Convergence Because Of Cancellation

So far, the tests for convergence have been applied to non negative terms only. Sometimes, a series converges, not because the terms of the series get small fast enough, but because of cancellation taking place between positive and negative terms. A discussion of this involves some simple algebra.

Let $\{a_n\}$ and $\{b_n\}$ be sequences and let

$$A_n \equiv \sum_{k=1}^{n} a_k,\ A_{-1} \equiv A_0 \equiv 0.$$

Then if $p < q$

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} b_n (A_n - A_{n-1}) = \sum_{n=p}^{q} b_n A_n - \sum_{n=p}^{q} b_n A_{n-1}$$

$$= \sum_{n=p}^{q} b_n A_n - \sum_{n=p-1}^{q-1} b_{n+1} A_n = b_q A_q - b_p A_{p-1} + \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) \tag{5.6}$$

This formula is called the partial summation formula.

Theorem 5.3.1 *(Dirichlet's test) Suppose A_n is bounded and* $\lim_{n\to\infty} b_n = 0$, *with $b_n \geq b_{n+1}$ for all n. Then*

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Proof: This follows quickly from Theorem 5.1.7. Indeed, letting $|A_n| \leq C$, and using the partial summation formula above along with the assumption that the b_n are decreasing,

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|b_q A_q - b_p A_{p-1} + \sum_{n=p}^{q-1} A_n (b_n - b_{n+1})\right|$$

$$\leq C(|b_q| + |b_p|) + C \sum_{n=p}^{q-1} (b_n - b_{n+1})$$

$$= C(|b_q| + |b_p|) + C(b_p - b_q)$$

and by assumption, this last expression is small whenever p and q are sufficiently large. This proves the theorem.

Definition 5.3.2 *If $b_n > 0$ for all n, a series of the form $\sum_k (-1)^k b_k$ or $\sum_k (-1)^{k-1} b_k$ is known as an alternating series.*

The following corollary is known as the alternating series test.

Corollary 5.3.3 *(alternating series test) If $\lim_{n\to\infty} b_n = 0$, with $b_n \geq b_{n+1}$, then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.*

Proof: Let $a_n = (-1)^n$. Then the partial sums of $\sum_n a_n$ are bounded and so Theorem 5.3.1 applies.

In the situation of Corollary 5.3.3 there is a convenient error estimate available.

Theorem 5.3.4 *Let $b_n > 0$ for all n such that $b_n \geq b_{n+1}$ for all n and $\lim_{n\to\infty} b_n = 0$ and consider either $\sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Then*

$$\left|\sum_{n=1}^{\infty} (-1)^n b_n - \sum_{n=1}^{N} (-1)^n b_n\right| \leq |b_{N+1}|,$$

$$\left|\sum_{n=1}^{\infty} (-1)^{n-1} b_n - \sum_{n=1}^{N} (-1)^{n-1} b_n\right| \leq |b_{N+1}|$$

See Problem 10 on Page 86 for an outline of the proof of this theorem along with another way to prove the alternating series test.

Example 5.3.5 *How many terms must I take in the sum, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}$ to be closer than $\frac{1}{10}$ to $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1}$?*

From Theorem 5.3.4, I need to find n such that $\frac{1}{n^2+1} \leq \frac{1}{10}$ and then $n-1$ is the desired value. Thus $n = 3$ and so

$$\left|\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2+1} - \sum_{n=1}^{2} (-1)^n \frac{1}{n^2+1}\right| \leq \frac{1}{10}$$

Definition 5.3.6 *A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. It is said to converge conditionally if $\sum |a_n|$ fails to converge but $\sum a_n$ converges.*

Thus the alternating series or more general Dirichlet test can determine convergence of series which converge conditionally.

5.3.2 Ratio And Root Tests

A favorite test for convergence is the ratio test. This is discussed next. It is at the other extreme from the alternating series test, being completely oblivious to any sort of cancellation. It only gives absolute convergence or spectacular divergence.

Theorem 5.3.7 *Suppose* $|a_n| > 0$ *for all* n *and suppose*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r.$$

Then

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{diverges if } r > 1 \\ \text{converges absolutely if } r < 1 \\ \text{test fails if } r = 1 \end{cases} .$$

Proof: Suppose $r < 1$. Then there exists n_1 such that if $n \geq n_1$, then

$$0 < \left| \frac{a_{n+1}}{a_n} \right| < R$$

where $r < R < 1$. Then

$$|a_{n+1}| < R\,|a_n|$$

for all such n. Therefore,

$$|a_{n_1+p}| < R\,|a_{n_1+p-1}| < R^2\,|a_{n_1+p-2}| < \cdots < R^p\,|a_{n_1}| \tag{5.7}$$

and so if $m > n$, then $|a_m| < R^{m-n_1}\,|a_{n_1}|$. By the comparison test and the theorem on geometric series, $\sum |a_n|$ converges. This proves the convergence part of the theorem.

To verify the divergence part, note that if $r > 1$, then 5.7 can be turned around for some $R > 1$. Showing $\lim_{n\to\infty} |a_n| = \infty$. Since the n^{th} term fails to converge to 0, it follows the series diverges.

To see the test fails if $r = 1$, consider $\sum n^{-1}$ and $\sum n^{-2}$. The first series diverges while the second one converges but in both cases, $r = 1$. (Be sure to check this last claim.)

The ratio test is very useful for many different examples but it is somewhat unsatisfactory mathematically. One reason for this is the assumption that $a_n > 0$, necessitated by the need to divide by a_n, and the other reason is the possibility that the limit might not exist. The next test, called the root test removes both of these objections. Before presenting this test, it is necessary to first prove the existence of the p^{th} root of any positive number.

Lemma 5.3.8 *Let* $\alpha > 0$ *be any nonnegative number and let* $p \in \mathbb{N}$. *Then* $\alpha^{1/p}$ *exists. This is the unique positive number which when raised to the* p^{th} *power gives* α.

Proof: If $\alpha = 0$ there is nothing to show. Assume then that $\alpha > 0$. First of all it follows by the binomial theorem that for x and y positive numbers,

$$\begin{aligned} y^p &= (x + y - x)^p = \sum_{k=0}^{p} \binom{p}{k} x^{p-k} (y - x)^k \\ &\geq x^p + p x^{p-1} (y - x) . \end{aligned} \tag{5.8}$$

Now let $x_1 > 0$ and define

$$x_{n+1} \equiv \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{1-p}. \tag{5.9}$$

(In case you wonder where this strange formula comes from, it is just the Newton Raphson method applied to the function $y = x^p - \alpha$. Of course this has not been discussed yet but you likely saw it in calculus. It is not important where it comes from, however.) In 5.8 let $x = x_n$ and let $y = x_{n+1}$ where x_{n+1} comes from 5.9. Then the right side of 5.8 equals

$$x_n^p + px_n^{p-1}\left(\frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{1-p} - x_n\right) = \alpha$$

and so $x_{n+1}^p \geq \alpha$ for all $n \geq 1$. Thus for $n > 1$,

$$\begin{aligned} x_{n+1} &= \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{1-p} \\ &= \frac{p-1}{p}x_n + \frac{\alpha}{p}\frac{x_n}{x_n^p} \\ &\leq \frac{p-1}{p}x_n + \frac{\alpha}{p}\frac{x_n}{\alpha} = x_n \end{aligned}$$

showing the sequence is decreasing. It is also bounded below by 0 and so by Theorem 4.9.7 and 4.4.11 the sequence converges to $x \geq 0$. I claim x cannot equal 0. If it did, you could pass to the limit in

$$x_{n+1}x_n^{p-1} = \frac{p-1}{p} + \frac{\alpha}{p}$$

and conclude

$$0 = \frac{p-1}{p} + \frac{\alpha}{p}$$

which is obviously false. Then passing to the limit yields a positive number, x such that

$$x = \frac{p-1}{p}x + \frac{\alpha}{p}\frac{x}{x^p}$$

which can hold if and only if $x^p = \alpha$. This proves the existence part of the lemma.

If $x_1^p = x_2^p$, then

$$0 = x_1^p - x_2^p = (x_1 - x_2)\left(\sum_{k=1}^{p-1} x_1^k x_2^{p-k}\right)$$

and so $x_1 = x_2$. This proves the lemma.

Theorem 5.3.9 *Suppose $|a_n|^{1/n} < R < 1$ for all n sufficiently large. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges absolutely.}$$

If there are infinitely many values of n such that $|a_n|^{1/n} \geq 1$, then

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Proof: Suppose first that $|a_n|^{1/n} < R < 1$ for all n sufficiently large. Say this holds for all $n \geq n_R$. Then for such n,

$$\sqrt[n]{|a_n|} < R.$$

Therefore, for such n,

$$|a_n| \leq R^n$$

and so the comparison test with a geometric series applies and gives absolute convergence as claimed.

Next suppose $|a_n|^{1/n} \geq 1$ for infinitely many values of n. Then for those values of n, $|a_n| \geq 1$ and so the series fails to converge by the n^{th} term test.

Stated more succinctly the condition for the root test is this: Let

$$r = \lim\sup_{n\to\infty} |a_n|^{1/n}$$

then

$$\sum_{k=m}^{\infty} a_k \begin{cases} \text{converges absolutely if } r < 1 \\ \text{test fails if } r = 1 \\ \text{diverges if } r > 1 \end{cases}$$

A special case occurs when the limit exists.

Corollary 5.3.10 *Suppose* $\lim_{n\to\infty} |a_n|^{1/n}$ *exists and equals* r. *Then*

$$\sum_{k=m}^{\infty} a_k \begin{cases} \textit{converges absolutely if } r < 1 \\ \textit{test fails if } r = 1 \\ \textit{diverges if } r > 1 \end{cases}$$

Proof: The first and last alternatives follow from Theorem 5.3.9. To see the test fails if $r = 1$, consider the two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ both of which have $r = 1$ but having different convergence properties.

5.4 Double Series

Sometimes it is required to consider double series which are of the form

$$\sum_{k=m}^{\infty}\sum_{j=m}^{\infty} a_{jk} \equiv \sum_{k=m}^{\infty}\left(\sum_{j=m}^{\infty} a_{jk}\right).$$

In other words, first sum on j yielding something which depends on k and then sum these. The major consideration for these double series is the question of when

$$\sum_{k=m}^{\infty}\sum_{j=m}^{\infty} a_{jk} = \sum_{j=m}^{\infty}\sum_{k=m}^{\infty} a_{jk}.$$

In other words, when does it make no difference which subscript is summed over first? In the case of finite sums there is no issue here. You can always write

$$\sum_{k=m}^{M}\sum_{j=m}^{N} a_{jk} = \sum_{j=m}^{N}\sum_{k=m}^{M} a_{jk}$$

because addition is commutative. However, there are limits involved with infinite sums and the interchange in order of summation involves taking limits in a different order. Therefore, it is not always true that it is permissible to interchange the two sums. A general rule of thumb is this: If something involves changing the order in which two limits are taken, you may not do it without agonizing over the question. In general, limits foul up algebra and also introduce things which are counter intuitive. Here is an example. This example is a little technical. It is placed here just to prove conclusively there is a question which needs to be considered.

Example 5.4.1 *Consider the following picture which depicts some of the ordered pairs (m,n) where m,n are positive integers.*

0•	0•	0•	0•	0•	c•	0•	-c•
0•	0•	0•	0•	c•	0•	-c•	0•
0•	0•	0•	c•	0•	-c•	0•	0•
0•	0•	c•	0•	-c•	0•	0•	0•
0•	c•	0•	-c•	0•	0•	0•	0•
b•	0•	-c•	0•	0•	0•	0•	0•
0•	a•	0•	0•	0•	0•	0•	0•

The numbers next to the point are the values of a_{mn}. You see $a_{nn} = 0$ for all n, $a_{21} = a, a_{12} = b, a_{mn} = c$ for (m,n) on the line $y = 1 + x$ whenever $m > 1$, and $a_{mn} = -c$ for all (m,n) on the line $y = x - 1$ whenever $m > 2$.

Then $\sum_{m=1}^{\infty} a_{mn} = a$ if $n = 1$, $\sum_{m=1}^{\infty} a_{mn} = b - c$ if $n = 2$ and if $n > 2, \sum_{m=1}^{\infty} a_{mn} = 0$. Therefore,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} = a + b - c.$$

Next observe that $\sum_{n=1}^{\infty} a_{mn} = b$ if $m = 1, \sum_{n=1}^{\infty} a_{mn} = a + c$ if $m = 2$, and $\sum_{n=1}^{\infty} a_{mn} = 0$ if $m > 2$. Therefore,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} = b + a + c$$

and so the two sums are different. Moreover, you can see that by assigning different values of a, b, and c, you can get an example for any two different numbers desired.

Don't become upset by this. It happens because, as indicated above, limits are taken in two different orders. An infinite sum always involves a limit and this illustrates why you must always remember this. This example in no way violates the commutative law of addition which has nothing to do with limits. However, it turns out that if $a_{ij} \geq 0$ for all i, j, then you can always interchange the order of summation. This is shown next and is based on the following lemma. First, some notation should be discussed.

Definition 5.4.2 *Let $f(a,b) \in [-\infty, \infty]$ for $a \in A$ and $b \in B$ where A, B are sets which means that $f(a,b)$ is either a number, ∞, or $-\infty$. The symbol, $+\infty$ is interpreted as a point out at the end of the number line which is larger than every real number. Of course there is no such number. That is why it is called ∞. The symbol, $-\infty$ is interpreted similarly. Then $\sup_{a\in A} f(a,b)$ means $\sup(S_b)$ where $S_b \equiv \{f(a,b) : a \in A\}$.*

Unlike limits, you can take the sup in different orders.

Lemma 5.4.3 *Let $f(a,b) \in [-\infty, \infty]$ for $a \in A$ and $b \in B$ where A, B are sets. Then*

$$\sup_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Proof: Note that for all a, b, $f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b)$ and therefore, for all a, $\sup_{b\in B} f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b)$. Therefore,

$$\sup_{a\in A}\sup_{b\in B} f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Repeat the same argument interchanging a and b, to get the conclusion of the lemma.

Lemma 5.4.4 *If $\{A_n\}$ is an increasing sequence in $[-\infty, \infty]$, then*

$$\sup\{A_n\} = \lim_{n\to\infty} A_n.$$

Proof: Let $\sup(\{A_n : n \in \mathbb{N}\}) = r$. In the first case, suppose $r < \infty$. Then letting $\varepsilon > 0$ be given, there exists n such that $A_n \in (r - \varepsilon, r]$. Since $\{A_n\}$ is increasing, it follows if $m > n$, then $r - \varepsilon < A_n \leq A_m \leq r$ and so $\lim_{n\to\infty} A_n = r$ as claimed. In the case where $r = \infty$, then if a is a real number, there exists n such that $A_n > a$. Since $\{A_k\}$ is increasing, it follows that if $m > n$, $A_m > a$. But this is what is meant by $\lim_{n\to\infty} A_n = \infty$. The other case is that $r = -\infty$. But in this case, $A_n = -\infty$ for all n and so $\lim_{n\to\infty} A_n = -\infty$.

Theorem 5.4.5 *Let $a_{ij} \geq 0$. Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof: First note there is no trouble in defining these sums because the a_{ij} are all nonnegative. If a sum diverges, it only diverges to ∞ and so ∞ is the value of the sum. Next note that

$$\sum_{j=r}^{\infty}\sum_{i=r}^{\infty} a_{ij} \geq \sup_n \sum_{j=r}^{\infty}\sum_{i=r}^{n} a_{ij}$$

because for all j,

$$\sum_{i=r}^{\infty} a_{ij} \geq \sum_{i=r}^{n} a_{ij}.$$

Therefore,

$$\sum_{j=r}^{\infty}\sum_{i=r}^{\infty} a_{ij} \geq \sup_n \sum_{j=r}^{\infty}\sum_{i=r}^{n} a_{ij} = \sup_n \lim_{m\to\infty} \sum_{j=r}^{m}\sum_{i=r}^{n} a_{ij}$$

$$= \sup_n \lim_{m\to\infty} \sum_{i=r}^{n}\sum_{j=r}^{m} a_{ij} = \sup_n \sum_{i=r}^{n} \lim_{m\to\infty} \sum_{j=r}^{m} a_{ij}$$

$$= \sup_n \sum_{i=r}^{n}\sum_{j=r}^{\infty} a_{ij} = \lim_{n\to\infty} \sum_{i=r}^{n}\sum_{j=r}^{\infty} a_{ij} = \sum_{i=r}^{\infty}\sum_{j=r}^{\infty} a_{ij}$$

Interchanging the i and j in the above argument proves the theorem.

The following is the fundamental result on double sums.

Theorem 5.4.6 *Let $a_{ij} \in \mathbb{F}$ and suppose*

$$\sum_{i=r}^{\infty}\sum_{j=r}^{\infty}|a_{ij}| < \infty \ .$$

Then

$$\sum_{i=r}^{\infty}\sum_{j=r}^{\infty}a_{ij} = \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}a_{ij}$$

and every infinite sum encountered in the above equation converges.

Proof: By Theorem 5.4.5

$$\sum_{j=r}^{\infty}\sum_{i=r}^{\infty}|a_{ij}| = \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}|a_{ij}| < \infty$$

Therefore, for each j, $\sum_{i=r}^{\infty}|a_{ij}| < \infty$ and for each i, $\sum_{j=r}^{\infty}|a_{ij}| < \infty$. By Theorem 5.1.9 on Page 72, $\sum_{i=r}^{\infty}a_{ij}$, $\sum_{j=r}^{\infty}a_{ij}$ both converge, the first one for every j and the second for every i. Also,

$$\sum_{j=r}^{\infty}\left|\sum_{i=r}^{\infty}a_{ij}\right| \leq \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}|a_{ij}| < \infty$$

and

$$\sum_{i=r}^{\infty}\left|\sum_{j=r}^{\infty}a_{ij}\right| \leq \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}|a_{ij}| < \infty$$

so by Theorem 5.1.9 again,

$$\sum_{j=r}^{\infty}\sum_{i=r}^{\infty}a_{ij}, \ \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}a_{ij}$$

both exist. It only remains to verify they are equal. By similar reasoning you can replace a_{ij} with $\operatorname{Re} a_{ij}$ or with $\operatorname{Im} a_{ij}$ in the above and the two sums will exist.

The real part of a finite sum of complex numbers equals the sum of the real parts. Then passing to a limit, it follows

$$\operatorname{Re}\sum_{j=r}^{\infty}\sum_{i=r}^{\infty}a_{ij} = \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}\operatorname{Re} a_{ij}$$

and similarly,

$$\operatorname{Im}\sum_{i=r}^{\infty}\sum_{j=r}^{\infty}a_{ij} = \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}\operatorname{Im} a_{ij}$$

Note $0 \leq (|a_{ij}| + \operatorname{Re} a_{ij}) \leq 2|a_{ij}|$. Therefore, by Theorem 5.4.5 and Theorem 5.1.5 on Page 70

$$\begin{aligned}\sum_{j=r}^{\infty}\sum_{i=r}^{\infty}|a_{ij}| + \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}\operatorname{Re} a_{ij} &= \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}(|a_{ij}| + \operatorname{Re} a_{ij}) \\ &= \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}(|a_{ij}| + \operatorname{Re} a_{ij}) \\ &= \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}|a_{ij}| + \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}\operatorname{Re} a_{ij} \\ &= \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}|a_{ij}| + \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}\operatorname{Re} a_{ij}\end{aligned}$$

and so

$$\operatorname{Re}\sum_{j=r}^{\infty}\sum_{i=r}^{\infty}a_{ij} = \sum_{j=r}^{\infty}\sum_{i=r}^{\infty}\operatorname{Re} a_{ij} = \sum_{i=r}^{\infty}\sum_{j=r}^{\infty}\operatorname{Re} a_{ij} = \operatorname{Re}\sum_{i=r}^{\infty}\sum_{j=r}^{\infty}a_{ij}$$

Similar reasoning applies to the imaginary parts. Since the real and imaginary parts of the two series are equal, it follows the two series are equal. This proves the theorem.

One of the most important applications of this theorem is to the problem of multiplication of series.

Definition 5.4.7 *Let $\sum_{i=r}^{\infty} a_i$ and $\sum_{i=r}^{\infty} b_i$ be two series. For $n \geq r$, define*

$$c_n \equiv \sum_{k=r}^{n} a_k b_{n-k+r}.$$

The series $\sum_{n=r}^{\infty} c_n$ is called the Cauchy product of the two series.

It isn't hard to see where this comes from. Formally write the following in the case $r = 0$:

$$(a_0 + a_1 + a_2 + a_3 \cdots)(b_0 + b_1 + b_2 + b_3 \cdots)$$

and start multiplying in the usual way. This yields

$$a_0 b_0 + (a_0 b_1 + b_0 a_1) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$

and you see the expressions in parentheses above are just the c_n for $n = 0, 1, 2, \cdots$. Therefore, it is reasonable to conjecture that

$$\sum_{i=r}^{\infty} a_i \sum_{j=r}^{\infty} b_j = \sum_{n=r}^{\infty} c_n$$

and of course there would be no problem with this in the case of finite sums but in the case of infinite sums, it is necessary to prove a theorem. The following is a special case of Merten's theorem.

Theorem 5.4.8 *Suppose $\sum_{i=r}^{\infty} a_i$ and $\sum_{j=r}^{\infty} b_j$ both converge absolutely*[1]*. Then*

$$\left(\sum_{i=r}^{\infty} a_i\right)\left(\sum_{j=r}^{\infty} b_j\right) = \sum_{n=r}^{\infty} c_n$$

[1] Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten's theorem and may be read in the 1974 book by Apostol listed in the bibliography.

where

$$c_n = \sum_{k=r}^{n} a_k b_{n-k+r}.$$

Proof: Let $p_{nk} = 1$ if $r \leq k \leq n$ and $p_{nk} = 0$ if $k > n$. Then

$$c_n = \sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r}.$$

Also,

$$\begin{aligned}
\sum_{k=r}^{\infty}\sum_{n=r}^{\infty} p_{nk} |a_k| |b_{n-k+r}| &= \sum_{k=r}^{\infty} |a_k| \sum_{n=r}^{\infty} p_{nk} |b_{n-k+r}| \\
&= \sum_{k=r}^{\infty} |a_k| \sum_{n=k}^{\infty} |b_{n-k+r}| \\
&= \sum_{k=r}^{\infty} |a_k| \sum_{n=k}^{\infty} \left|b_{n-(k-r)}\right| \\
&= \sum_{k=r}^{\infty} |a_k| \sum_{m=r}^{\infty} |b_m| < \infty.
\end{aligned}$$

Therefore, by Theorem 5.4.6

$$\begin{aligned}
\sum_{n=r}^{\infty} c_n &= \sum_{n=r}^{\infty}\sum_{k=r}^{n} a_k b_{n-k+r} = \sum_{n=r}^{\infty}\sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r} \\
&= \sum_{k=r}^{\infty} a_k \sum_{n=r}^{\infty} p_{nk} b_{n-k+r} = \sum_{k=r}^{\infty} a_k \sum_{n=k}^{\infty} b_{n-k+r} \\
&= \sum_{k=r}^{\infty} a_k \sum_{m=r}^{\infty} b_m
\end{aligned}$$

and this proves the theorem.

5.5 Exercises

1. Determine whether the following series converge absolutely, conditionally, or not at all and give reasons for your answers.

   (a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2+n+1}}$

   (b) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

   (c) $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$

   (d) $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n!)^2}$

   (e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+2}$

   (f) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)^n$

   (g) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)^{n^2}$

2. Determine whether the following series converge absolutely, conditionally, or not at all and give reasons for your answers.

   (a) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n+n}{n2^n}$
   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n+n}{n^2 2^n}$
   (c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$
   (d) $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n!}$
   (e) $\sum_{n=1}^{\infty} (-1)^n \frac{n^{100}}{1.01^n}$
   (f) $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n^3}$
   (g) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$
   (h) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n!}$
   (i) $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^{100}}$

3. Find the exact values of the following infinite series if they converge.

   (a) $\sum_{k=3}^{\infty} \frac{1}{k(k-2)}$
   (b) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$
   (c) $\sum_{k=3}^{\infty} \frac{1}{(k+1)(k-2)}$
   (d) $\sum_{k=1}^{N} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$

4. Suppose $\sum_{n=1}^{\infty} a_n$ converges absolutely. Can the same thing be said about $\sum_{n=1}^{\infty} a_n^2$? Explain.

5. A person says a series converges conditionally by the ratio test. Explain why his statement is total nonsense.

6. A person says a series diverges by the alternating series test. Explain why his statement is total nonsense.

7. Find a series which diverges using one test but converges using another if possible. If this is not possible, tell why.

8. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, can you conclude the sum, $\sum_{n=1}^{\infty} a_n b_n$ converges?

9. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and b_n is bounded, can you conclude $\sum_{n=1}^{\infty} a_n b_n$ converges? What if it is only the case that $\sum_{n=1}^{\infty} a_n$ converges?

10. Prove Theorem 5.3.4. **Hint:** For $\sum_{k=1}^{\infty} (-1)^n b_n$, show the odd partial sums are all at least as small as $\sum_{k=1}^{\infty} (-1)^n b_n$ and are increasing while the even partial sums are at least as large as $\sum_{k=1}^{\infty} (-1)^n b_n$ and are decreasing. Use this to give another proof of the alternating series test. If you have trouble, see most standard calculus books.

11. Use Theorem 5.3.4 in the following alternating series to tell how large n must be so that $\left|\sum_{k=1}^{\infty} (-1)^k a_k - \sum_{k=1}^{n} (-1)^k a_k\right|$ is no larger than the given number.

    (a) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$, .001

(b) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$, .001

(c) $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{\sqrt{k}}$, .001

12. Consider the series $\sum_{k=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}$. Show this series converges and so it makes sense to write $\left(\sum_{k=0}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}\right)^2$. What about the Cauchy product of this series? Does it even converge? What does this mean about using algebra on infinite sums as though they were finite sums?

13. Verify Theorem 5.4.8 on the two series $\sum_{k=0}^{\infty} 2^{-k}$ and $\sum_{k=0}^{\infty} 3^{-k}$.

14. You can define infinite series of complex numbers in exactly the same way as infinite series of real numbers. That is $w = \sum_{k=1}^{\infty} z_k$ means: For every $\varepsilon > 0$ there exists N such that if $n \geq N$, then $\left|w - \sum_{k=1}^{N} z_k\right| < \varepsilon$. Here the absolute value is the one which applies to complex numbers. That is, $|a + ib| = \sqrt{a^2 + b^2}$. Show that if $\{a_n\}$ is a decreasing sequence of nonnegative numbers with the property that $\lim_{n\to\infty} a_n = 0$ and if ω is any complex number which is not equal to 1 but which satisfies $|\omega| = 1$, then $\sum_{k=1}^{\infty} \omega^n a_n$ must converge. Note a sequence of complex numbers, $\{a_n + ib_n\}$ converges to $a + ib$ if and only if $a_n \to a$ and $b_n \to b$. See Problem 6 on Page 53. There are quite a few things in this problem you should think about.

15. Suppose $\lim_{k\to\infty} s_k = s$. Show it follows $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = s$.

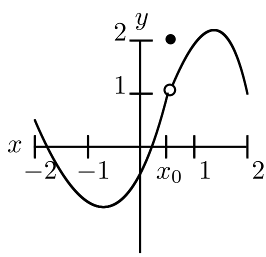
16. Using Problem 15 show that if $\sum_{j=1}^{\infty} \frac{a_j}{j}$ converges, then it follows

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j = 0.$$

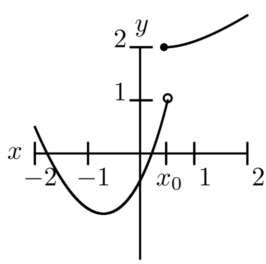
Continuous Functions

The concept of function is far too general to be useful in calculus. There are various ways to restrict the concept in order to study something interesting and the types of restrictions considered depend very much on what you find interesting. In Calculus, the most fundamental restriction made is to assume the functions are continuous. Continuous functions are those in which a sufficiently small change in x results in a small change in $f(x)$. They rule out things which could never happen physically. For example, it is not possible for a car to jump from one point to another instantly. Making this restriction precise turns out to be surprisingly difficult although many of the most important theorems about continuous functions seem intuitively clear.

Before giving the careful mathematical definitions, here are examples of graphs of functions which are not continuous at the point x_0.



You see, there is a hole in the picture of the graph of this function and instead of filling in the hole with the appropriate value, $f(x_0)$ is too large. This is called a removable discontinuity because the problem can be fixed by redefining the function at the point x_0. Here is another example.



You see from this picture that there is no way to get rid of the jump in the graph of this function by simply redefining the value of the function at x_0. That is why it is called a nonremovable discontinuity or jump discontinuity. Now that pictures have been given of what it is desired to eliminate, it is time to give the precise definition.

The definition which follows, due to Cauchy[1] and Weierstrass[2] is the precise way to exclude the sort of behavior described above and all statements about continuous functions must ultimately rest on this definition from now on.

Definition 6.0.1 *A function $f : D(f) \subseteq \mathbb{F} \to \mathbb{F}$ is continuous at $x \in D(f)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $y \in D(f)$ and*

$$|y - x| < \delta$$

it follows that

$$|f(x) - f(y)| < \varepsilon.$$

A function, f is continuous if it is continuous at every point of $D(f)$.

In sloppy English this definition says roughly the following: A function, f is continuous at x when it is possible to make $f(y)$ as close as desired to $f(x)$ provided y is taken close enough to x. In fact this statement in words is pretty much the way Cauchy described it. The completely rigorous definition above is due to Weierstrass. This definition does indeed rule out the sorts of graphs drawn above. Consider the second nonremovable discontinuity.

[1] Augustin Louis Cauchy 1789-1857 was the son of a lawyer who was married to an aristocrat. He was born in France just after the fall of the Bastille and his family fled the reign of terror and hid in the countryside till it was over. Cauchy was educated at first by his father who taught him Greek and Latin. Eventually Cauchy learned many languages. He was also a good Catholic.

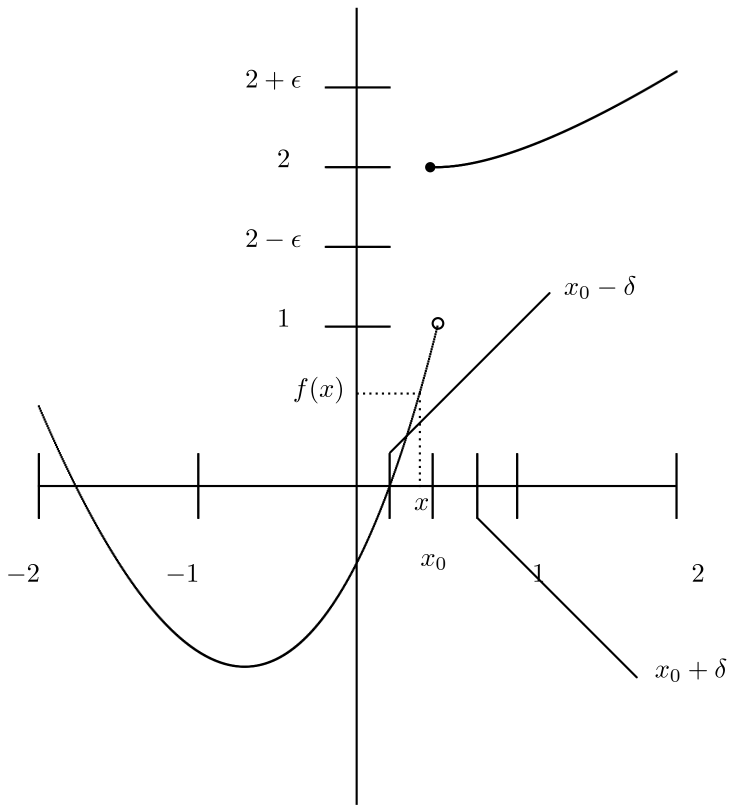
After the reign of terror, the family returned to Paris and Cauchy studied at the university to be an engineer but became a mathematician although he made fundamental contributions to physics and engineering. Cauchy was one of the most prolific mathematicians who ever lived. He wrote several hundred papers which fill 24 volumes. He also did research on many topics in mechanics and physics including elasticity, optics and astronomy. More than anyone else, Cauchy invented the subject of complex analysis. He is also credited with giving the first rigorous definition of continuity.

He married in 1818 and lived for 12 years with his wife and two daughters in Paris till the revolution of 1830. Cauchy refused to take the oath of allegience to the new ruler and ended up leaving his family and going into exile for 8 years.

Notwithstanding his great achievments he was not known as a popular teacher.

[2] Wilhelm Theodor Weierstrass 1815-1897 brought calculus to essentially the state it is in now. When he was a secondary school teacher, he wrote a paper which was so profound that he was granted a doctor's degree. He made fundamental contributions to partial differential equations, complex analysis, calculus of variations, and many other topics. He also discovered some pathological examples such as space filling curves. Cauchy gave the definition in words and Weierstrass, somewhat later produced the totally rigorous ε δ definition presented here. The need for rigor in the subject of calculus was only realized over a long period of time.

The removable discontinuity case is similar.



For the ε shown you can see from the picture that no matter how small you take δ, there will be points, x, between $x_0-\delta$ and x_0 where $f(x)<2-\varepsilon$. In particular, for these values of x, $|f(x)-f(x_0)|>\varepsilon$. Therefore, the definition of continuity given above excludes the situation in which there is a jump in the function. Similar reasoning shows it excludes the removable discontinuity case as well. There are many ways a function can fail to be continuous and it is impossible to list them all by drawing pictures. This is why it is so important to use the definition. The other thing to notice is that the concept of continuity as described in the definition is a point property. That is to say it is a property which a function may or may not have at a single point. Here is an example.

Example 6.0.2 *Let*

$$f(x)=\begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$$

This function is continuous at $x=0$ and nowhere else.

To verify the assertion about the above function, first show it is not continuous at x if $x\neq 0$. Take such an x and let $\varepsilon=|x|/2$. Now let $\delta>0$ be completely arbitrary. In the interval, $(x-\delta,x+\delta)$ there are rational numbers, y_1 such that $|y_1|>|x|$ and irrational numbers, y_2. Thus $|f(y_1)-f(y_2)|=|y_1|>|x|$. If f were continuous at x, there would exist $\delta>0$ such that for every point, $y\in(x-\delta,x+\delta)$, $|f(y)-f(x)|<\varepsilon$. But then, letting y_1

and y_2 be as just described,

$$\begin{aligned} |x| < |y_1| &= |f(y_1) - f(y_2)| \\ &\leq |f(y_1) - f(x)| + |f(x) - f(y_2)| < 2\varepsilon = |x|, \end{aligned}$$

which is a contradiction. Since a contradiction is obtained by assuming that f is continuous at x, it must be concluded that f is not continuous there. To see f is continuous at 0, let $\varepsilon > 0$ be given and let $\delta = \varepsilon$. Then if $|y - 0| < \delta = \varepsilon$, Then

$$\begin{aligned} |f(y) - f(0)| &= 0 \text{ if } y \text{ is irrational} \\ |f(y) - f(0)| &= |y| < \varepsilon \text{ if } y \text{ is rational.} \end{aligned}$$

either way, whenever $|y - 0| < \delta$, it follows $|f(y) - f(0)| < \varepsilon$ and so f is continuous at $x = 0$. How did I know to let $\delta = \varepsilon$? That is a very good question. The choice of δ for a particular ε is usually arrived at by using intuition, the actual ε δ argument reduces to a verification that the intuition was correct. Here is another example.

Example 6.0.3 *Show the function,* $f(x) = -5x + 10$ *is continuous at* $x = -3$.

To do this, note first that $f(-3) = 25$ and it is desired to verify the conditions for continuity. Consider the following.

$$|-5x + 10 - (25)| = 5|x - (-3)|.$$

This allows one to find a suitable δ. If $\varepsilon > 0$ is given, let $0 < \delta \leq \frac{1}{5}\varepsilon$. Then if $0 < |x - (-3)| < \delta$, it follows from this inequality that

$$|-5x + 10 - (25)| = 5|x - (-3)| < 5\frac{1}{5}\varepsilon = \varepsilon.$$

Sometimes the determination of δ in the verification of continuity can be a little more involved. Here is another example.

Example 6.0.4 *Show the function,* $f(x) = \sqrt{2x + 12}$ *is continuous at* $x = 5$.

First note $f(5) = \sqrt{22}$. Now consider:

$$\left|\sqrt{2x+12} - \sqrt{22}\right| = \left|\frac{2x + 12 - 22}{\sqrt{2x+12} + \sqrt{22}}\right|$$

$$= \frac{2}{\sqrt{2x+12} + \sqrt{22}}|x - 5| \leq \frac{1}{11}\sqrt{22}|x - 5|$$

whenever $|x - 5| < 1$ because for such x, $\sqrt{2x+12} > 0$. Now let $\varepsilon > 0$ be given. Choose δ such that $0 < \delta \leq \min\left(1, \frac{\varepsilon\sqrt{22}}{2}\right)$. Then if $|x - 5| < \delta$, all the inequalities above hold and

$$\left|\sqrt{2x+12} - \sqrt{22}\right| \leq \frac{2}{\sqrt{22}}|x - 5| < \frac{2}{\sqrt{22}}\frac{\varepsilon\sqrt{22}}{2} = \varepsilon.$$

Exercise 6.0.5 *Show* $f(x) = -3x^2 + 7$ *is continuous at* $x = 7$.

First observe $f(7) = -140$. Now

$$\left|-3x^2 + 7 - (-140)\right| = 3|x+7||x-7| \leq 3(|x| + 7)|x-7|.$$

If $|x-7| < 1$, it follows from the version of the triangle inequality which states $||s| - |t|| \leq |s-t|$ that $|x| < 1 + 7$. Therefore, if $|x-7| < 1$, it follows that

$$\begin{aligned}\left|-3x^2 + 7 - (-140)\right| &\leq 3((1+7)+7)|x-7| \\ &= 3(1+27)|x-7| = 84|x-7|.\end{aligned}$$

Now let $\varepsilon > 0$ be given. Choose δ such that $0 < \delta \leq \min\left(1, \frac{\varepsilon}{84}\right)$. Then for $|x-7| < \delta$, it follows

$$\left|-3x^2 + 7 - (-140)\right| \leq 84|x-7| < 84\left(\frac{\varepsilon}{84}\right) = \varepsilon.$$

The following is a useful theorem which can remove the need to constantly use the ε, δ definition given above.

Theorem 6.0.6 *The following assertions are valid*

1. *The function, $af + bg$ is continuous at x when f, g are continuous at $x \in D(f) \cap D(g)$ and $a, b \in \mathbb{F}$.*
2. *If and f and g are each continuous at x, then fg is continuous at x. If, in addition to this, $g(x) \neq 0$, then f/g is continuous at x.*
3. *If f is continuous at x, $f(x) \in D(g) \subseteq \mathbb{F}$, and g is continuous at $f(x)$,then $g \circ f$ is continuous at x.*
4. *The function $f : \mathbb{F} \to \mathbb{R}$, given by $f(x) = |x|$ is continuous.*

Proof: First consider 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|x-y| < \delta_1$, it follows $|f(x) - f(y)| < \frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\delta_2 > 0$ such that whenever $|x-y| < \delta_2$, it follows that $|g(x) - g(y)| < \frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|x-y| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$|af(x) + bf(x) - (ag(y) + bg(y))|$$

$$\begin{aligned}&\leq |a||f(x) - f(y)| + |b||g(x) - g(y)| \\ &< |a|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right) + |b|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right) < \varepsilon.\end{aligned}$$

Now consider 2.) There exists $\delta_1 > 0$ such that if $|y-x| < \delta_1$, then $|f(x) - f(y)| < 1$. Therefore, for such y,

$$|f(y)| < 1 + |f(x)|.$$

It follows that for such y,

$$|fg(x) - fg(y)| \leq |f(x)g(x) - g(x)f(y)| + |g(x)f(y) - f(y)g(y)|$$

$$\begin{aligned}&\leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \\ &\leq (1 + |g(x)| + |f(y)|)[|g(x) - g(y)| + |f(x) - f(y)|] \\ &\leq (2 + |g(x)| + |f(x)|)[|g(x) - g(y)| + |f(x) - f(y)|]\end{aligned}$$

Now let $\varepsilon > 0$ be given. There exists δ_2 such that if $|x-y| < \delta_2$, then

$$|g(x) - g(y)| < \frac{\varepsilon}{2(1+|g(x)|+|f(y)|)},$$

and there exists δ_3 such that if $|x-y| < \delta_3$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2(2+|g(x)|+|f(x)|)}$$

Now let $0 < \delta \leq \min(\delta_1, \delta_2, \delta_3)$. Then if $|x-y| < \delta$, all the above hold at once and so

$$|fg(x) - fg(y)| \leq$$

$$\begin{aligned}
&(2+|g(x)|+|f(x)|)\,[|g(x)-g(y)|+|f(x)-f(y)|] \\
&< (2+|g(x)|+|f(x)|)\left(\frac{\varepsilon}{2(2+|g(x)|+|f(x)|)} + \frac{\varepsilon}{2(2+|g(x)|+|f(x)|)}\right) = \varepsilon.
\end{aligned}$$

This proves the first part of 2.) To obtain the second part, let δ_1 be as described above and let $\delta_0 > 0$ be such that for $|x-y| < \delta_0$,

$$|g(x) - g(y)| < |g(x)|/2$$

and so by the triangle inequality,

$$-|g(x)|/2 \leq |g(y)| - |g(x)| \leq |g(x)|/2$$

which implies $|g(y)| \geq |g(x)|/2$, and $|g(y)| < 3|g(x)|/2$.

Then if $|x-y| < \min(\delta_0, \delta_1)$,

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right| &= \left|\frac{f(x)g(y) - f(y)g(x)}{g(x)g(y)}\right| \\
&\leq \frac{|f(x)g(y) - f(y)g(x)|}{\left(\frac{|g(x)|^2}{2}\right)} \\
&= \frac{2|f(x)g(y) - f(y)g(x)|}{|g(x)|^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{|g(x)|^2}[|f(x)g(y) - f(y)g(y) + f(y)g(y) - f(y)g(x)|] \\
&\leq \frac{2}{|g(x)|^2}[|g(y)||f(x)-f(y)| + |f(y)||g(y)-g(x)|] \\
&\leq \frac{2}{|g(x)|^2}\left[\frac{3}{2}|g(x)||f(x)-f(y)| + (1+|f(x)|)|g(y)-g(x)|\right] \\
&\leq \frac{2}{|g(x)|^2}(1+2|f(x)|+2|g(x)|)[|f(x)-f(y)| + |g(y)-g(x)|] \\
&\equiv M[|f(x)-f(y)| + |g(y)-g(x)|]
\end{aligned}$$

where M is defined by

$$M \equiv \frac{2}{|g(x)|^2}(1+2|f(x)|+2|g(x)|)$$

Now let δ_2 be such that if $|x-y| < \delta_2$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2} M^{-1}$$

and let δ_3 be such that if $|x-y| < \delta_3$, then

$$|g(y) - g(x)| < \frac{\varepsilon}{2} M^{-1}.$$

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|x-y| < \delta$, everything holds and

$$\left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| \leq M [|f(x) - f(y)| + |g(y) - g(x)|]$$

$$< M \left[\frac{\varepsilon}{2} M^{-1} + \frac{\varepsilon}{2} M^{-1} \right] = \varepsilon.$$

This completes the proof of the second part of 2.)

Note that in these proofs no effort is made to find some sort of "best" δ. The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now consider 3.). If f is continuous at x, $f(x) \in D(g) \subseteq \mathbb{F}$, and g is continuous at $f(x)$,then $g \circ f$ is continuous at x. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$ such that if $|y-f(x)| < \eta$ and $y \in D(g)$, it follows that $|g(y) - g(f(x))| < \varepsilon$. From continuity of f at x, there exists $\delta > 0$ such that if $|x-z| < \delta$ and $z \in D(f)$, then $|f(z) - f(x)| < \eta$. Then if $|x-z| < \delta$ and $z \in D(g \circ f) \subseteq D(f)$, all the above hold and so

$$|g(f(z)) - g(f(x))| < \varepsilon.$$

This proves part 3.)

To verify part 4.), let $\varepsilon > 0$ be given and let $\delta = \varepsilon$. Then if $|x-y| < \delta$, the triangle inequality implies

$$|f(x) - f(y)| = ||x| - |y||$$
$$\leq |x-y| < \delta = \varepsilon.$$

This proves part 4.) and completes the proof of the theorem.

6.1 Continuity And The Limit Of A Sequence

There is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 6.1.1 *A function $f : D(f) \to \mathbb{F}$ is continuous at $x \in D(f)$ if and only if, whenever $x_n \to x$ with $x_n \in D(f)$, it follows $f(x_n) \to f(x)$.*

Proof: Suppose first that f is continuous at x and let $x_n \to x$. Let $\varepsilon > 0$ be given. By continuity, there exists $\delta > 0$ such that if $|y - x| < \delta$, then $|f(x) - f(y)| < \varepsilon$. However, there exists n_δ such that if $n \geq n_\delta$, then $|x_n - x| < \delta$ and so for all n this large,

$$|f(x) - f(x_n)| < \varepsilon$$

which shows $f(x_n) \to f(x)$.

Now suppose the condition about taking convergent sequences to convergent sequences holds at x. Suppose f fails to be continuous at x. Then there exists $\varepsilon > 0$ and $x_n \in D(f)$ such that $|x - x_n| < \frac{1}{n}$, yet

$$|f(x) - f(x_n)| \geq \varepsilon.$$

But this is clearly a contradiction because, although $x_n \to x$, $f(x_n)$ fails to converge to $f(x)$. It follows f must be continuous after all. This proves the theorem.

Theorem 6.1.2 *Suppose $f : D(f) \to \mathbb{R}$ is continuous at $x \in D(f)$ and suppose $f(x_n) \leq l\ (\geq l)$ where $\{x_n\}$ is a sequence of points of $D(f)$ which converges to x. Then $f(x) \leq l\ (\geq l)$.*

Proof: Since $f(x_n) \leq l$ and f is continuous at x, it follows from Theorem 4.4.11 and Theorem 6.1.1,

$$f(x) = \lim_{n\to\infty} f(x_n) \leq l.$$

The other case is entirely similar. This proves the theorem.

6.2 Exercises

1. Let $f(x) = 2x + 7$. Show f is continuous at every point x. **Hint:** You need to let $\varepsilon > 0$ be given. In this case, you should try $\delta \leq \varepsilon/2$. Note that if one δ works in the definition, then so does any smaller δ.

2. Suppose $D(f) = [0,1] \cup \{9\}$ and $f(x) = x$ on $[0,1]$ while $f(9) = 5$. Is f continuous at the point, 9? Use whichever definition of continuity you like.

3. Let $f(x) = x^2 + 1$. Show f is continuous at $x = 3$. **Hint:**

$$\begin{aligned} |f(x) - f(3)| &= \left|x^2 + 1 - (9 + 1)\right| \\ &= |x + 3|\,|x - 3|. \end{aligned}$$

   Thus if $|x - 3| < 1$, it follows from the triangle inequality, $|x| < 1 + 3 = 4$ and so

$$|f(x) - f(3)| < 4\,|x - 3|.$$

   Now try to complete the argument by letting $\delta \leq \min(1, \varepsilon/4)$. The symbol, min means to take the minimum of the two numbers in the parenthesis.

4. Let $f(x) = 2x^2 + 1$. Show f is continuous at $x = 1$.

5. Let $f(x) = x^2 + 2x$. Show f is continuous at $x = 2$. Then show it is continuous at every point.

6. Let $f(x) = |2x + 3|$. Show f is continuous at every point. **Hint:** Review the two versions of the triangle inequality for absolute values.

7. Let $f(x) = \frac{1}{x^2+1}$. Show f is continuous at every value of x.

8. If $x \in \mathbb{R}$, show there exists a sequence of rational numbers, $\{x_n\}$ such that $x_n \to x$ and a sequence of irrational numbers, $\{x'_n\}$ such that $x'_n \to x$. Now consider the following function.

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$$

   Show using the sequential version of continuity in Theorem 6.1.1 that f is discontinuous at every point.

9. If $x \in \mathbb{R}$, show there exists a sequence of rational numbers, $\{x_n\}$ such that $x_n \to x$ and a sequence of irrational numbers, $\{x'_n\}$ such that $x'_n \to x$. Now consider the following function.
$$f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}.$$
Show using the sequential version of continuity in Theorem 6.1.1 that f is continuous at 0 and nowhere else.

10. Use the sequential definition of continuity described above to give an easy proof of Theorem 6.0.6.

11. Let $f(x) = \sqrt{x}$ show f is continuous at every value of x in its domain. For now, assume $\sqrt{x}$ exists for all positive x. **Hint:** You might want to make use of the identity,
$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$
at some point in your argument.

12. Using Theorem 6.0.6, show all polynomials are continuous and that a rational function is continuous at every point of its domain. **Hint:** First show the function given as $f(x) = x$ is continuous and then use the Theorem 6.0.6. What about the case where x can be in $\mathbb{F}$? Does the same conclusion hold?

13. Let $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ and consider $g(x) = f(x)\left(x - x^3\right)$. Determine where g is continuous and explain your answer.

14. Suppose f is any function whose domain is the integers. Thus $D(f) = \mathbb{Z}$, the set of whole numbers, $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$. Then f is continuous. Why? **Hint:** In the definition of continuity, what if you let $\delta = \frac{1}{4}$? Would this δ work for a given $\varepsilon > 0$? This shows that the idea that a continuous function is one for which you can draw the graph without taking the pencil off the paper is a lot of nonsense.

15. Give an example of a function, f which is not continuous at some point but $|f|$ is continuous at that point.

16. Find two functions which fail to be continuous but whose product is continuous.

17. Find two functions which fail to be continuous but whose sum is continuous.

18. Find two functions which fail to be continuous but whose quotient is continuous.

19. Suppose f is an increasing function defined on $[a, b]$. Show f must be continuous at all but a countable set of points. **Hint:** Explain why every discontinuity of f is a jump discontinuity and
$$f(x-) \equiv \lim_{y \to x-} f(y) \leq f(x) \leq f(x+) \equiv \lim_{y \to x+} f(y)$$
with $f(x+) > f(x-)$. Now each of these intervals $(f(x-), f(x+))$ at a point, x where a discontinuity happens has positive length and they are disjoint. Furthermore, they have to all fit in $[f(a), f(b)]$. How many of them can there be which have length at least $1/n$?

20. Suppose f is a function defined on $\mathbb{R}$ and f is continuous at 0. Suppose also that $f(x + y) = f(x) + f(y)$. Show that if this is so, then f must be continuous at every value of $x \in \mathbb{R}$. Next show that for every rational number, r, $f(r) = rf(1)$. Finally explain why $f(r) = rf(1)$ for every r a real number. **Hint:** To do this last part, you need to use the density of the rational numbers and continuity of f.

6.3 The Extreme Values Theorem

The extreme values theorem says continuous functions achieve their maximum and minimum provided they are defined on a sequentially compact set.

Lemma 6.3.1 *Let $K \subseteq \mathbb{F}$ be sequentially compact and let $f : K \to \mathbb{R}$ be continuous. Then f is bounded. That is there exist numbers, m and M such that for all $x \in [a, b]$,*

$$m \leq f(x) \leq M.$$

Proof: Suppose f is not bounded above. Then there exists $\{x_n\}$, a sequence of points in K such that $f(x_n) \geq n$. Since K is sequentially compact, there is a subsequence $\{x_{n_k}\}$ and a point in K, x such that $x_{n_k} \to x$. Then by Theorem 6.1.1, $f(x_{n_k}) \to f(x) \in \mathbb{R}$ and this is a contradiction to $f(x_{n_k}) > n_k$. Thus f must be bounded above. Similarly, f must be bounded below. This proves the lemma.

Example 6.3.2 *Let $f(x) = 1/x$ for $x \in (0, 1)$.*

Clearly, f is not bounded. Does this violate the conclusion of the above lemma? It does not because the end points of the interval involved are not in the interval. The same function defined on $[.000001, 1)$ would have been bounded although in this case the boundedness of the function would not follow from the above lemma because it fails to include the right endpoint.

The next theorem is known as the max min theorem or extreme value theorem.

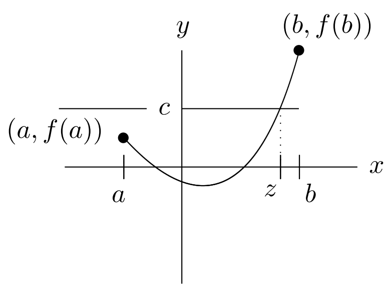
Theorem 6.3.3 *Let $K \subseteq \mathbb{F}$ be sequentially compact and let $f : K \to \mathbb{R}$ be continuous. Then f achieves its maximum and its minimum on K. This means there exist, $x_1, x_2 \in K$ such that for all $x \in K$,*

$$f(x_1) \leq f(x) \leq f(x_2).$$

Proof: By Lemma 6.3.1 $f(K)$ has a least upper bound, M. If for all $x \in K$, $f(x) \neq M$, then by Theorem 6.0.6, the function, $g(x) \equiv (M - f(x))^{-1} = \frac{1}{M - f(x)}$ is continuous on K. Since M is the least upper bound of $f(K)$ there exist points, $x \in K$ such that $(M - f(x))$ is as small as desired. Consequently, g is not bounded above, contrary to Lemma 6.3.1. Therefore, there must exist some $x \in K$ such that $f(x) = M$. This proves f achieves its maximum. The argument for the minimum is similar. Alternatively, you could consider the function $h(x) = M - f(x)$. Then use what was just proved to conclude h achieves its maximum at some point, x_1. Thus $h(x_1) \geq h(x)$ for all $x \in I$ and so $M - f(x_1) \geq M - f(x)$ for all $x \in I$ which implies $f(x_1) \leq f(x)$ for all $x \in I$. This proves the theorem.

6.4 The Intermediate Value Theorem

The next big theorem is called the intermediate value theorem and the following picture illustrates its conclusion. It gives the existence of a certain point.



You see in the picture there is a horizontal line, $y = c$ and a continuous function which starts off less than c at the point a and ends up greater than c at point b. The intermediate value theorem says there is some point between a and b shown in the picture as z such that the value of the function at this point equals c. It may seem this is obvious but without completeness the conclusion of the theorem cannot be drawn. Nevertheless, the above picture makes this theorem very easy to believe.

Next here is a proof of the intermediate value theorem.

Theorem 6.4.1 *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and suppose $f(a) < c < f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof: Let $d = \frac{a+b}{2}$ and consider the intervals $[a, d]$ and $[d, b]$. If $f(d) \geq c$, then on $[a, d]$, the function is $\leq c$ at one end point and $\geq c$ at the other. On the other hand, if $f(d) \leq c$, then on $[d, b]$, $f \geq 0$ at one end point and ≤ 0 at the other. Pick the interval on which f has values which are at least as large as c and values no larger than c. Now consider that interval, divide it in half as was done for the original interval and argue that on one of these smaller intervals, the function has values at least as large as c and values no larger than c. Continue in this way. Next apply the nested interval lemma to get x in all these intervals. In the n^{th} interval, let x_n, y_n be points of this interval such that $f(x_n) \leq c$, $f(y_n) \geq c$. Now $|x_n - x| \leq (b - a) 2^{-n}$ and $|y_n - x| \leq (b - a) 2^{-n}$ and so $x_n \to x$ and $y_n \to x$. Therefore,

$$f(x) - c = \lim_{n\to\infty} (f(x_n) - c) \leq 0$$

while

$$f(x) - c = \lim_{n\to\infty} (f(y_n) - c) \geq 0.$$

Consequently $f(x) = c$ and this proves the theorem. The last step follows from Theorem 6.1.1.

Here is another proof of the intermediate value theorem.

Theorem 6.4.2 *Suppose $f : [a, b] \to \mathbb{R}$ is continuous and suppose $f(a) < c < f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof: Since $f(a) < c$, the set, S defined as

$$S \equiv \{y \in [a, b] : f(t) < c \text{ for all } t \leq y\}$$

is nonempty. In particular $a \in S$. Also, since $S \subseteq [a, b]$, S is bounded above and so there exists a least upper bound, $x \leq b$. I claim $f(x) \leq c$. Suppose $f(x) > c$. Then letting $\varepsilon \equiv f(x) - c$, there exists $\delta > 0$ such that if $|y - x| < \delta$ and $y \in [a, b]$ then

$$f(x) - f(y) \leq |f(x) - f(y)| < \varepsilon \equiv f(x) - c$$

which implies that for such y, $f(y) > c$. Therefore, $x > a$ and for all $y \in [a, b] \cap (x - \delta, x)$, it follows $f(y) > c$ and this violates the definition of x as the least upper bound of S. In particular those $y \in [a, b] \cap (x - \delta, x)$ are upper bounds to S and these are smaller than x. Thus $f(x) \leq c$. Next suppose $f(x) < c$ which implies $x < b$. Then letting $\varepsilon \equiv c - f(x)$ there exists $\delta > 0$ such that for $|y - x| < \delta$ and $y \in [a, b]$,

$$f(y) - f(x) \leq |f(x) - f(y)| < \varepsilon = c - f(x)$$

and so for $y \in (x, x + \delta) \cap [a, b]$, it follows $f(y) < c$. But for $y < x$, $f(y) \leq c$ and so x fails to be an upper bound for S. Therefore, it must be the case that $f(x) = c$ and x is not equal to either a or b. This proves the theorem.

Lemma 6.4.3 *Let $\phi : [a,b] \to \mathbb{R}$ be a continuous function and suppose ϕ is $1-1$ on (a,b). Then ϕ is either strictly increasing or strictly decreasing on $[a,b]$.*

Proof: First it is shown that ϕ is either strictly increasing or strictly decreasing on (a,b).

If ϕ is not strictly decreasing on (a,b), then there exists $x_1 < y_1$, $x_1, y_1 \in (a,b)$ such that

$$(\phi(y_1) - \phi(x_1))(y_1 - x_1) > 0.$$

If for some other pair of points, $x_2 < y_2$ with $x_2, y_2 \in (a,b)$, the above inequality does not hold, then since ϕ is $1-1$,

$$(\phi(y_2) - \phi(x_2))(y_2 - x_2) < 0.$$

Let $x_t \equiv tx_1 + (1-t)x_2$ and $y_t \equiv ty_1 + (1-t)y_2$. Then $x_t < y_t$ for all $t \in [0,1]$ because

$$tx_1 \leq ty_1 \text{ and } (1-t)x_2 \leq (1-t)y_2$$

with strict inequality holding for at least one of these inequalities since not both t and $(1-t)$ can equal zero. Now define

$$h(t) \equiv (\phi(y_t) - \phi(x_t))(y_t - x_t).$$

Since h is continuous and $h(0) < 0$, while $h(1) > 0$, there exists $t \in (0,1)$ such that $h(t) = 0$. Therefore, both x_t and y_t are points of (a,b) and $\phi(y_t) - \phi(x_t) = 0$ contradicting the assumption that ϕ is one to one. It follows ϕ is either strictly increasing or strictly decreasing on (a,b).

This property of being either strictly increasing or strictly decreasing on (a,b) carries over to $[a,b]$ by the continuity of ϕ. Suppose ϕ is strictly increasing on (a,b), a similar argument holding for ϕ strictly decreasing on (a,b). If $x > a$, then pick $y \in (a,x)$ and from the above, $\phi(y) < \phi(x)$. Now by continuity of ϕ at a,

$$\phi(a) = \lim_{x \to a+} \phi(z) \leq \phi(y) < \phi(x).$$

Therefore, $\phi(a) < \phi(x)$ whenever $x \in (a,b)$. Similarly $\phi(b) > \phi(x)$ for all $x \in (a,b)$. This proves the lemma.

Corollary 6.4.4 *Let $f : (a,b) \to \mathbb{R}$ be one to one and continuous. Then $f(a,b)$ is an open interval, (c,d) and $f^{-1} : (c,d) \to (a,b)$ is continuous.*

Proof: Since f is either strictly increasing or strictly decreasing, it follows that $f(a,b)$ is an open interval, (c,d). Assume f is decreasing. Now let $x \in (a,b)$. Why is f^{-1} is continuous at $f(x)$? Since f is decreasing, if $f(x) < f(y)$, then $y \equiv f^{-1}(f(y)) < x \equiv f^{-1}(f(x))$ and so f^{-1} is also decreasing. Let $\varepsilon > 0$ be given. Let $\varepsilon > \eta > 0$ and $(x-\eta, x+\eta) \subseteq (a,b)$. Then $f(x) \in (f(x+\eta), f(x-\eta))$. Let $\delta = \min(f(x) - f(x+\eta), f(x-\eta) - f(x))$. Then if

$$|f(z) - f(x)| < \delta,$$

it follows

$$z \equiv f^{-1}(f(z)) \in (x-\eta, x+\eta) \subseteq (x-\varepsilon, x+\varepsilon)$$

so

$$\left|f^{-1}(f(z)) - x\right| = \left|f^{-1}(f(z)) - f^{-1}(f(x))\right| < \varepsilon.$$

This proves the theorem in the case where f is strictly decreasing. The case where f is increasing is similar.

6.5 Exercises

1. Suppose $f : K \to \mathbb{R}$ is continuous where K is sequentially compact subset of $\mathbb{F}$. Show f achieves its minimum on K in the following way: Let $\lambda \equiv \inf \{f(x) : x \in [a, b]\}$. Explain why there exists a sequence, $\{x_n\}$ such that $\lambda = \lim_{n\to\infty} f(x_n)$. Adapt in an appropriate manner if $\lambda = -\infty$. Then since K is sequentially compact, there is a subsequence, $\{x_{n_k}\}$ such that it converges to $x \in K$. Then by continuity, $f(x) = \lambda$ so $\lambda \neq -\infty$ and f achieves its minimum. Next show that f achieves its maximum on K.

2. Give an example of a continuous function defined on $(0, 1)$ which does not achieve its maximum on $(0, 1)$.

3. Give an example of a continuous function defined on $(0, 1)$ which is bounded but which does not achieve either its maximum or its minimum.

4. Give an example of a discontinuous function defined on $[0, 1]$ which is bounded but does not achieve either its maximum or its minimum.

5. Give an example of a continuous function defined on $[0, 1) \cup (1, 2]$ which is positive at 2, negative at 0 but is not equal to zero for any value of x.

6. Let $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ where $a, b, c, d,$ and e are numbers. Show there exists real x such that $f(x) = 0$.

7. Give an example of a function which is one to one but neither strictly increasing nor strictly decreasing.

8. Show that the function, $f(x) = x^n - a$, where n is a positive integer and a is a number, is continuous.

9. Use the intermediate value theorem on the function, $f(x) = x^7 - 8$ to show $\sqrt[7]{8}$ must exist. State and prove a general theorem about n^{th} roots of positive numbers.

10. Prove $\sqrt{2}$ is irrational. **Hint:** Suppose $\sqrt{2} = p/q$ where p, q are positive integers and the fraction is in lowest terms. Then $2q^2 = p^2$ and so p^2 is even. Explain why $p = 2r$ so p must be even. Next argue q must be even.

11. Let $f(x) = x - \sqrt{2}$ for $x \in \mathbb{Q}$, the rational numbers. Show that even though $f(0) < 0$ and $f(2) > 0$, there is no point in $\mathbb{Q}$ where $f(x) = 0$. Does this contradict the intermediate value theorem? Explain.

12. It has been known since the time of Pythagoras that $\sqrt{2}$ is irrational. If you throw out all the irrational numbers, show that the conclusion of the intermediate value theorem could no longer be obtained. That is, show there exists a function which starts off less than zero and ends up larger than zero and yet there is no number where the function equals zero. **Hint:** Try $f(x) = x^2 - 2$. You supply the details.

13. A circular hula hoop lies partly in the shade and partly in the hot sun. Show there exist two points on the hula hoop which are at opposite sides of the hoop which have the same temperature. **Hint:** Imagine this is a circle and points are located by specifying their angle, θ from a fixed diameter. Then letting $T(\theta)$ be the temperature in the hoop, $T(\theta + 2\pi) = T(\theta)$. You need to have $T(\theta) = T(\theta + \pi)$ for some θ. Assume T is a continuous function of θ.

14. A car starts off on a long trip with a full tank of gas. The driver intends to drive the car till it runs out of gas. Show that at some time the number of miles the car has gone exactly equals the number of gallons of gas in the tank.

15. Suppose f is a continuous function defined on $[0,1]$ which maps $[0,1]$ into $[0,1]$. Show there exists $x \in [0,1]$ such that $x = f(x)$. **Hint:** Consider $h(x) \equiv x - f(x)$ and the intermediate value theorem.

6.6 Uniform Continuity

There is a theorem about the integral of a continuous function which requires the notion of uniform continuity. This is discussed in this section. Consider the function $f(x) = \frac{1}{x}$ for $x \in (0,1)$. This is a continuous function because, by Theorem 6.0.6, it is continuous at every point of $(0,1)$. However, for a given $\varepsilon > 0$, the δ needed in the ε, δ definition of continuity becomes very small as x gets close to 0. The notion of uniform continuity involves being able to choose a single δ which works on the whole domain of f. Here is the definition.

Definition 6.6.1 *Let f be a function. Then f is uniformly continuous if for every $\varepsilon > 0$, there exists a δ* ***depending only on*** *ε such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.*

It is an amazing fact that under certain conditions continuity implies uniform continuity.

Theorem 6.6.2 *Let $f : K \to \mathbb{F}$ be continuous where K is a sequentially compact set in $\mathbb{F}$. Then f is uniformly continuous on K.*

Proof: If this is not true, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a pair of points, x_δ and y_δ such that even though $|x_\delta - y_\delta| < \delta$, $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. Taking a succession of values for δ equal to $1, 1/2, 1/3, \cdots$, and letting the exceptional pair of points for $\delta = 1/n$ be denoted by x_n and y_n,

$$|x_n - y_n| < \frac{1}{n}, |f(x_n) - f(y_n)| \geq \varepsilon.$$

Now since K is sequentially compact, there exists a subsequence, $\{x_{n_k}\}$ such that $x_{n_k} \to z \in K$. Now $n_k \geq k$ and so

$$|x_{n_k} - y_{n_k}| < \frac{1}{k}.$$

Consequently, $y_{n_k} \to z$ also. (x_{n_k} is like a person walking toward a certain point and y_{n_k} is like a dog on a leash which is constantly getting shorter. Obviously y_{n_k} must also move toward the point also. You should give a precise proof of what is needed here.) By continuity of f and Theorem 6.1.2,

$$0 = |f(z) - f(z)| = \lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon,$$

an obvious contradiction. Therefore, the theorem must be true.

The following corollary follows from this theorem and Theorem 4.7.2.

Corollary 6.6.3 *Suppose K is a closed interval, $[a,b]$, a set of the form $[a,b] + i[c,d]$, or*

$$D(z,r) \equiv \{w : |z-w| \leq r\}.$$

Then f is uniformly continuous.

6.7 Exercises

1. A function f is Lipschitz continuous or just Lipschitz for short if there exists a constant, K such that
$$|f(x) - f(y)| \leq K|x - y|$$
for all $x, y \in D$. Show every Lipschitz function is uniformly continuous.

2. If $|x_n - y_n| \to 0$ and $x_n \to z$, show that $y_n \to z$ also. This was used in the proof of Theorem 6.6.2.

3. Consider $f : (1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Show f is uniformly continuous even though the set on which f is defined is not sequentially compact.

4. If f is uniformly continuous, does it follow that $|f|$ is also uniformly continuous? If $|f|$ is uniformly continuous does it follow that f is uniformly continuous? Answer the same questions with "uniformly continuous" replaced with "continuous". Explain why.

5. Suppose f is a function defined on D and $\lambda \equiv \inf\{f(x) : x \in D\}$. A sequence $\{x_n\}$ of points of D is called a minimizing sequence if $\lim_{n\to\infty} f(x_n) = \lambda$. A maximizing sequence is defined analogously. Show that minimizing sequences and maximizing sequences always exist. Now let K be a sequentially compact set and $f : K \to \mathbb{R}$. Show that f achieves both its maximum and its minimum on K by considering directly minimizing and maximizing sequences. **Hint:** Let $M \equiv \sup\{f(x) : x \in K\}$. Argue there exists a sequence, $\{x_n\} \subseteq K$ such that $f(x_n) \to M$. Now use sequential compactness to get a subsequence, $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x \in K$ and use the continuity of f to verify that $f(x) = M$. Incidentally, this shows f is bounded on K as well. A similar argument works to give the part about achieving the minimum.

6. Let $f : D \to \mathbb{R}$ be a function. This function is said to be lower semicontinuous[3] at $x \in D$ if for any sequence $\{x_n\} \subseteq D$ which converges to x it follows
$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$
Suppose D is sequentially compact and f is lower semicontinuous at every point of D. Show that then f achieves its minimum on D.

7. Let $f : D \to \mathbb{R}$ be a function. This function is said to be upper semicontinuous at $x \in D$ if for any sequence $\{x_n\} \subseteq D$ which converges to x it follows
$$f(x) \geq \limsup_{n\to\infty} f(x_n).$$
Suppose D is sequentially compact and f is upper semicontinuous at every point of D. Show that then f achieves its maximum on D.

8. Show that a real valued function is continuous if and only if it is both upper and lower semicontinuous.

9. Show that a real valued lower semicontinuous function defined on a sequentially compact set achieves its minimum and that an upper semicontinuous function defined on a sequentially compact set achieves its maximum.

[3]The notion of lower semicontinuity is very important for functions which are defined on infinite dimensional sets. In more general settings, one formulates the concept differently.

10. Give an example of a lower semicontinuous function which is not continuous and an example of an upper semicontinuous function which is not continuous.
11. Suppose $\{f_\alpha : \alpha \in \Lambda\}$ is a collection of continuous functions. Let
$$F(x) \equiv \inf \{f_\alpha(x) : \alpha \in \Lambda\}$$
Show F is an upper semicontinuous function. Next let
$$G(x) \equiv \sup \{f_\alpha(x) : \alpha \in \Lambda\}$$
Show G is a lower semicontinuous function.
12. Let f be a function. $\operatorname{epi}(f)$ is defined as
$$\{(x, y) : y \geq f(x)\}.$$
It is called the epigraph of f. We say $\operatorname{epi}(f)$ is closed if whenever $(x_n, y_n) \in \operatorname{epi}(f)$ and $x_n \to x$ and $y_n \to y$, it follows $(x, y) \in \operatorname{epi}(f)$. Show f is lower semicontinuous if and only if $\operatorname{epi}(f)$ is closed. What would be the corresponding result equivalent to upper semicontinuous?

6.8 Sequences And Series Of Functions

When you understand sequences and series of numbers it is easy to consider sequences and series of functions.

Definition 6.8.1 *A sequence of functions is a map defined on $\mathbb{N}$ or some set of integers larger than or equal to a given integer, m which has values which are functions. It is written in the form $\{f_n\}_{n=m}^{\infty}$ where f_n is a function. It is assumed also that the domain of all these functions is the same.*

In the above, where do the functions have values? Are they real valued functions? Are they complex valued functions? Are they functions which have values in $\mathbb{R}^n$? It turns out it does not matter very much and the same definition holds. However, if you like, you can think of them as having values in $\mathbb{F}$. This is the main case of interest here.

Example 6.8.2 *Suppose $f_n(x) = x^n$ for $x \in [0, 1]$.*

Definition 6.8.3 *Let $\{f_n\}$ be a sequence of functions. Then the sequence converges pointwise to a function f if for all $x \in D$, the domain of the functions in the sequence,*
$$f(x) = \lim_{n \to \infty} f_n(x)$$
This is always the definition regardless of where the f_n have their values.

Thus you consider for each $x \in D$ the sequence of numbers $\{f_n(x)\}$ and if this sequence converges for each $x \in D$, the thing it converges to is called $f(x)$.

Example 6.8.4 *In Example 6.8.2 find* $\lim_{n \to \infty} f_n$.

For $x \in [0, 1), \lim_{n \to \infty} x^n = f_n(x) = 0$. At $x = 1, f_n(1) = 1$ for all n so $\lim_{n \to \infty} f_n(1) = 1$. Therefore, this sequence of functions converges pointwise to the function $f(x)$ given by $f(x) = 0$ if $0 \leq x < 1$ and $f(1) = 1$.

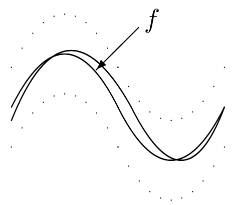
Pointwise convergence is a very inferior thing but sometimes it is all you can get. It's undesirability is illustrated by the preceding example. The limit function is not continuous although each f_n is continuous. The next definition pertains to a superior sort of convergence called uniform convergence.

Definition 6.8.5 *Let $\{f_n\}$ be a sequence of functions defined on D. Then $\{f_n\}$ is said to converge uniformly to f if it converges pointwise to f and for every $\varepsilon > 0$ there exists N such that for all $n \geq N$*

$$|f(x) - f_n(x)| < \varepsilon$$

for all $x \in D$.

The following picture illustrates the above definition.



The dotted lines define sort of a tube centered about the graph of f and the graph of the function f_n fits in this tube.

The reason uniform convergence is desirable is that it drags continuity along with it and imparts this property to the limit function.

Theorem 6.8.6 *Let $\{f_n\}$ be a sequence of continuous functions defined on D and suppose this sequence converges uniformly to f. Then f is also continuous on D. If each f_n is uniformly continuous on D, then f is also uniformly continuous on D.*

Proof: Let $\varepsilon > 0$ be given and pick $z \in D$. By uniform convergence, there exists N such that if $n > N$, then for all $x \in D$,

$$|f(x) - f_n(x)| < \varepsilon/3. \tag{6.1}$$

Pick such an n. By assumption, f_n is continuous at z. Therefore, there exists $\delta > 0$ such that if $|z - x| < \delta$ then

$$|f_n(x) - f_n(z)| < \varepsilon/3.$$

It follows that for $|x - z| < \delta$,

$$\begin{aligned} |f(x) - f(z)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(z)| + |f_n(z) - f(z)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

which shows that since ε was arbitrary, f is continuous at z.

In the case where each f_n is uniformly continuous, and using the same f_n for which 6.1 holds, there exists a $\delta > 0$ such that if $|y - z| < \delta$, then

$$|f_n(z) - f_n(y)| < \varepsilon/3.$$

Then for $|y - z| < \delta$,

$$\begin{aligned} |f(y) - f(z)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(z)| + |f_n(z) - f(z)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

This shows uniform continuity of f. This proves the theorem.

Definition 6.8.7 *Let $\{f_n\}$ be a sequence of functions defined on D. Then the sequence is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists N such that whenever $m, n \geq N$,*

$$|f_m(x) - f_n(x)| < \varepsilon$$

for all $x \in D$.

Then the following theorem follows easily.

Theorem 6.8.8 *Let $\{f_n\}$ be a uniformly Cauchy sequence $\mathbb{F}$ valued functions defined on D. Then there exists f defined on D such that $\{f_n\}$ converges uniformly to f.*

Proof: For each $x \in D, \{f_n(x)\}$ is a Cauchy sequence. Therefore, it converges to some number because of completeness of $\mathbb{F}$. Denote by $f(x)$ this number. Let $\varepsilon > 0$ be given and let N be such that if $n, m \geq N$,

$$|f_m(x) - f_n(x)| < \varepsilon/2$$

for all $x \in D$. Then for any $x \in D$, pick $n \geq N$ and it follows from Theorem 4.4.11

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \varepsilon/2 < \varepsilon.$$

This proves the theorem.

Corollary 6.8.9 *Let $\{f_n\}$ be a uniformly Cauchy sequence of functions continuous on D. Then there exists f defined on D such that $\{f_n\}$ converges uniformly to f and f is continuous. Also, if each f_n is uniformly continuous, then so is f.*

Proof: This follows from Theorem 6.8.8 and Theorem 6.8.6. This proves the Corollary. Here is one more fairly obvious theorem.

Theorem 6.8.10 *Let $\{f_n\}$ be a sequence of functions defined on D. Then it converges pointwise if and only if the sequence $\{f_n(x)\}$ is a Cauchy sequence for every $x \in D$. It converges uniformly if and only if $\{f_n\}$ is a uniformly Cauchy sequence.*

Proof: If the sequence converges pointwise, then by Theorem 4.9.3 the sequence $\{f_n(x)\}$ is a Cauchy sequence for each $x \in D$. Conversely, if $\{f_n(x)\}$ is a Cauchy sequence for each $x \in D$, then since f_n has values in $\mathbb{F}$, and $\mathbb{F}$ is complete, it follows the sequence $\{f_n(x)\}$ converges for each $x \in D$. (Recall that completeness is the same as saying every Cauchy sequence converges.)

Now suppose $\{f_n\}$ is uniformly Cauchy. Then from Theorem 6.8.8 there exists f such that $\{f_n\}$ converges uniformly on D to f. Conversely, if $\{f_n\}$ converges uniformly to f on D, then if $\varepsilon > 0$ is given, there exists N such that if $n \geq N$,

$$|f(x) - f_n(x)| < \varepsilon/2$$

for every $x \in D$. Then if $m, n \geq N$ and $x \in D$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\{f_n\}$ is uniformly Cauchy.

As before, once you understand sequences, it is no problem to consider series.

Definition 6.8.11 *Let $\{f_n\}$ be a sequence of functions defined on D. Then*

$$\left(\sum_{k=1}^{\infty} f_k\right)(x) \equiv \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) \tag{6.2}$$

whenever the limit exists. Thus there is a new function denoted by

$$\sum_{k=1}^{\infty} f_k \tag{6.3}$$

and its value at x is given by the limit of the sequence of partial sums in 6.2. If for all $x \in D$, the limit in 6.2 exists, then 6.3 is said to converge pointwise. $\sum_{k=1}^{\infty} f_k$ is said to converge uniformly on D if the sequence of partial sums,

$$\left\{\sum_{k=1}^{n} f_k\right\}$$

converges uniformly.If the indices for the functions start at some other value than 1, you make the obvious modification to the above definition as was done earlier with series of numbers.

Theorem 6.8.12 *Let $\{f_n\}$ be a sequence of functions defined on D. The series $\sum_{k=1}^{\infty} f_k$ converges pointwise if and only if for each $\varepsilon > 0$ and $x \in D$, there exists $N_{\varepsilon,x}$ which may depend on x as well as ε such that when $q > p \geq N_{\varepsilon,x}$,*

$$\left|\sum_{k=p}^{q} f_k(x)\right| < \varepsilon$$

The series $\sum_{k=1}^{\infty} f_k$ converges uniformly on D if for every $\varepsilon > 0$ there exists N_ε such that if $q > p \geq N$ then

$$\left|\sum_{k=p}^{q} f_k(x)\right| < \varepsilon \tag{6.4}$$

for all $x \in D$.

Proof: The first part follows from Theorem 5.1.7. The second part follows from observing the condition is equivalent to the sequence of partial sums forming a uniformly Cauchy sequence and then by Theorem 6.8.10, these partial sums converge uniformly to a function which is the definition of $\sum_{k=1}^{\infty} f_k$. This proves the theorem.

Is there an easy way to recognize when 6.4 happens? Yes, there is. It is called the Weierstrass M test.

Theorem 6.8.13 *Let $\{f_n\}$ be a sequence of functions defined on D. Suppose there exists M_n such that $\sup\{|f_n(x)| : x \in D\} < M_n$ and $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D.*

Proof: Let $z \in D$. Then letting $m < n$

$$\left|\sum_{k=1}^{n} f_k(z) - \sum_{k=1}^{m} f_k(z)\right| \leq \sum_{k=m+1}^{n} |f_k(z)| \leq \sum_{k=m+1}^{\infty} M_k < \varepsilon$$

whenever m is large enough because of the assumption that $\sum_{n=1}^{\infty} M_n$ converges. Therefore, the sequence of partial sums is uniformly Cauchy on D and therefore, converges uniformly to $\sum_{k=1}^{\infty} f_k$ on D. This proves the theorem.

Theorem 6.8.14 *If $\{f_n\}$ is a sequence of continuous functions defined on D and $\sum_{k=1}^{\infty} f_k$ converges uniformly, then the function, $\sum_{k=1}^{\infty} f_k$ must also be continuous.*

Proof: This follows from Theorem 6.8.6 applied to the sequence of partial sums of the above series which is assumed to converge uniformly to the function, $\sum_{k=1}^{\infty} f_k$.

6.9 Sequences Of Polynomials, Weierstrass Approximation

It turns out that if f is a continuous real valued function defined on an interval, $[a,b]$ then there exists a sequence of polynomials, $\{p_n\}$ such that the sequence converges uniformly to f on $[a,b]$. I will first show this is true for the interval $[0,1]$ and then verify it is true on any closed and bounded interval. First here is a little lemma which is interesting for its own sake in probability. It is actually an estimate for the variance of a binomial distribution.

Lemma 6.9.1 *The following estimate holds for $x \in [0,1]$ and $m \geq 2$.*

$$\sum_{k=0}^{m} \binom{m}{k} (k-mx)^2 x^k (1-x)^{m-k} \leq \frac{1}{4} m$$

Proof: First of all, from the binomial theorem

$$\begin{aligned}
&\sum_{k=0}^{m} \binom{m}{k} k x^k (1-x)^{m-k} \\
= &\sum_{k=1}^{m} \binom{m}{k} k x^k (1-x)^{m-k} \\
= &\ mx \sum_{k=1}^{m} \frac{(m-1)!}{(k-1)!\,(m-k)!} x^{k-1} (1-x)^{m-k} \\
= &\ mx \sum_{k=1}^{m} \frac{(m-1)!}{(k-1)!\,(m-1-k)!} x^{k} (1-x)^{m-1-k} \\
= &\ mx\,(1+1-x)^{m-1} = mx.
\end{aligned}$$

Next, using what was just shown and the binomial theorem again,

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k} k^2 x^k (1-x)^{m-k} &= \sum_{k=1}^{m} \binom{m}{k} k(k-1) x^k (1-x)^{m-k} \\
&\quad + \sum_{k=0}^{m} \binom{m}{k} k x^k (1-x)^{m-k}
\end{aligned}$$

$$= \sum_{k=2}^{m} \binom{m}{k} k(k-1) x^k (1-x)^{m-k} + mx$$

$$= \sum_{k=0}^{m-2} \binom{m}{k+2} (k+2)(k+1) x^{k+2} (1-x)^{m-2-k} + mx$$

$$\begin{aligned}
&= m(m-1)\sum_{k=0}^{m-2}\binom{m-2}{k}x^{k+2}(1-x)^{m-2-k}+mx \\
&= x^2 m(m-1)\sum_{k=0}^{m-2}\binom{m-2}{k}x^{k}(1-x)^{m-2-k}+mx \\
&= x^2 m(m-1)+mx = x^2m^2-x^2m+mx
\end{aligned}$$

It follows

$$\sum_{k=0}^{m}\binom{m}{k}(k-mx)^2 x^k(1-x)^{m-k}$$

$$=\sum_{k=0}^{m}\binom{m}{k}\left(k^2-2kmx+x^2m^2\right)x^k(1-x)^{m-k}$$

and from what was just shown along with the binomial theorem again, this equals

$$x^2m^2-x^2m+mx-2mx(mx)+x^2m^2=-x^2m+mx=\frac{m}{4}-m\left(x-\frac{1}{2}\right)^2.$$

Thus the expression is maximized when $x=1/2$ and yields $m/4$ in this case. This proves the lemma.

Now let f be a continuous function defined on $[0,1]$. Let p_n be the polynomial defined by

$$p_n(x)\equiv\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}. \tag{6.5}$$

Theorem 6.9.2 *The sequence of polynomials in 6.5 converges uniformly to f on $[0,1]$.*

Proof: By the binomial theorem,

$$f(x)=f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}f(x)x^k(1-x)^{n-k}$$

and so by the triangle inequality

$$|f(x)-p_n(x)|\leq\sum_{k=0}^{n}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f(x)\right|x^k(1-x)^{n-k}$$

At this point you bread the sum into two pieces, those values of k such that k/n is close to x and those values for k such that k/n is not so close to x. Thus

$$\begin{aligned}
|f(x)-p_n(x)| \leq& \sum_{|x-(k/n)|<\delta}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f(x)\right|x^k(1-x)^{n-k} \\
&+\sum_{|x-(k/n)|\geq\delta}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-f(x)\right|x^k(1-x)^{n-k}
\end{aligned} \tag{6.6}$$

where δ is a positive number chosen in an auspicious manner about to be described. Since f is continuous on $[0,1]$, it follows from Theorems 4.7.2 and 6.6.2 that f is uniformly continuous. Therefore, letting $\varepsilon>0$, there exists $\delta>0$ such that if $|x-y|<\delta$, then

$|f(x) - f(y)| < \varepsilon/2$. This is the auspicious choice for δ. Also, by Lemma 6.3.1 $|f(x)|$ for $x \in [0,1]$ is bounded by some number M. Thus 6.6 implies that for $x \in [0,1]$,

$$\begin{aligned}
|f(x) - p_n(x)| &\leq \sum_{|x-(k/n)|<\delta} \binom{n}{k} \frac{\varepsilon}{2} x^k (1-x)^{n-k} \\
&\quad + 2M \sum_{|nx-k|\geq n\delta} \binom{n}{k} x^k (1-x)^{n-k} \\
&\leq \frac{\varepsilon}{2} + 2M \sum_{|nx-k|\geq n\delta} \binom{n}{k} \frac{(k-nx)^2}{n^2\delta^2} x^k (1-x)^{n-k} \\
&\leq \frac{\varepsilon}{2} + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^k (1-x)^{n-k}
\end{aligned}$$

Now by Lemma 6.9.1 there is an estimate for the last sum. Using this estimate yields for all $x \in [0,1]$,

$$|f(x) - p_n(x)| \leq \frac{\varepsilon}{2} + \frac{2M}{n^2\delta^2}\frac{n}{4} = \frac{\varepsilon}{2} + \frac{M}{2n\delta^2}.$$

Therefore, whenever n is sufficiently large that

$$\frac{M}{2n\delta^2} < \frac{\varepsilon}{2},$$

it follows that for all $x \in [0,1]$,

$$|f(x) - p_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the theorem.

Now this theorem has been done, it is easy to extend to continuous functions defined on $[a,b]$. This yields the celebrated Weierstrass approximation theorem.

Theorem 6.9.3 *Suppose f is a continuous function defined on $[a,b]$. Then there exists a sequence of polynomials, $\{p_n\}$ which converges uniformly to f on $[a,b]$.*

Proof: For $t \in [0,1]$, let $h(t) = a + (b-a)t$. Thus h maps $[0,1]$ one to one and onto $[a,b]$. Thus $f \circ h$ is a continuous function defined on $[0,1]$. It follows there exists a sequence of polynomials $\{p_n\}$ defined on $[0,1]$ which converges uniformly to $f \circ h$ on $[0,1]$. Thus for every $\varepsilon > 0$ there exists N_ε such that if $n \geq N_\varepsilon$, then for all $t \in [0,1]$,

$$|f \circ h(t) - p_n(t)| < \varepsilon.$$

However, h is onto and one to one and so for all $x \in [a,b]$,

$$\left|f(x) - p_n\left(h^{-1}(x)\right)\right| < \varepsilon.$$

Now note that the function $x \rightarrow p_n\left(h^{-1}(x)\right)$ is a polynomial because

$$h^{-1}(x) = \frac{x-a}{b-a}.$$

More specifically, if

$$p_n(t) = \sum_{k=0}^{m} a_k t^k$$

it follows

$$p_n\left(h^{-1}(x)\right) = \sum_{k=0}^{m} a_k \left(\frac{x-a}{b-a}\right)^k$$

which is clearly another polynomial. This proves the theorem.

6.10 Exercises

1. Suppose $\{f_n\}$ is a sequence of decreasing positive functions defined on $[0, \infty)$ which converges pointwise to 0 for every $x \in [0, \infty)$. Can it be concluded that this sequence converges uniformly to 0 on $[0, \infty)$? Now replace $[0, \infty)$ with $(0, \infty)$. What can be said in this case assuming pointwise convergence still holds?

2. If $\{f_n\}$ and $\{g_n\}$ are sequences of functions defined on D which converge uniformly, show that if a, b are constants, then $af_n + bg_n$ also converges uniformly. If there exists a constant, M such that $|f_n(x)|, |g_n(x)| < M$ for all n and for all $x \in D$, show $\{f_n g_n\}$ converges uniformly. Let $f_n(x) \equiv 1/x$ for $x \in (0, 1)$ and let $g_n(x) \equiv (n-1)/n$. Show $\{f_n\}$ converges uniformly on $(0, 1)$ and $\{g_n\}$ converges uniformly but $\{f_n g_n\}$ fails to converge uniformly.

3. Show that if $x > 0$, $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges uniformly on any interval of finite length.

4. Let $x \geq 0$ and consider the sequence $\left\{\left(1 + \frac{x}{n}\right)^n\right\}$. Show this is an increasing sequence and is bounded above by $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

5. Show for every x, y real, $\sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}$ converges and equals
$$\sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

6. Consider the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Show this series converges uniformly on any interval of the form $[-M, M]$.

7. Formulate a theorem for series of functions which will allow you to conclude the infinite series is uniformly continuous based on reasonable assumptions about the functions in the sum.

8. Find an example of a sequence of continuous functions such that each function is nonnegative and each function has a maximum value equal to 1 but the sequence of functions converges to 0 pointwise on $(0, \infty)$.

9. Suppose $\{f_n\}$ is a sequence of real valued functions which converges uniformly to a continuous function f. Can it be concluded the functions f_n are continuous? Explain.

10. Let $h(x)$ be a bounded continuous function. Show the function $f(x) = \sum_{n=1}^{\infty} \frac{h(nx)}{n^2}$ is continuous.

11. Here is an awful function. Recalling $\mathbb{Q}$ is the rational numbers,
$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$
Show this function is discontinuous at every point of $\mathbb{R}$. Nevertheless, this horrible function is the pointwise limit of a sequence of uniformly continuous functions. Explain why this is so.

12. Let S be a any countable subset of $\mathbb{R}$. Show there exists a function, f defined on $\mathbb{R}$ which is discontinuous at every point of S but continuous everywhere else. **Hint:** This is real easy if you do the right thing. It involves Theorem 6.8.14 and the Weierstrass M test.

13. By Theorem 6.9.3 there exists a sequence of polynomials converging uniformly to $f(x) = |x|$ on the interval $[-1,1]$. Show there exists a sequence of polynomials, $\{p_n\}$ converging uniformly to f on $[-1,1]$ which has the additional property that for all $n, p_n(0) = 0$.

14. If f is any continuous function defined on $[a,b]$, show there exists a series of the form $\sum_{k=1}^{\infty} p_k$, where each p_k is a polynomial, which converges uniformly to f on $[a,b]$. **Hint:** You should use the Weierstrass approximation theorem to obtain a sequence of polynomials. Then arrange it so the limit of this sequence is an infinite sum.

15. Sometimes a series may converge uniformly without the Weierstrass M test being applicable. Show
$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$
converges uniformly on $[0,1]$ but does not converge absolutely for any $x \in \mathbb{R}$. To do this, it might help to use the partial summation formula, 5.6.

The Derivative

Some functions have them and some don't. Some have them at some points and not at others. This chapter is on the derivative. Functions which have derivatives are somehow better than those which don't. To begin with it is necessary to discuss the concept of a limit of a function. This is a harder concept than continuity and it is also harder than the concept of the limit of a sequence or series although that is similar. One cannot make any rational sense of the concept of derivative without an understanding of limits of a function.

7.1 Limit Of A Function

In this section, functions will be defined on some subset of $\mathbb{R}$ having values in $\mathbb{F}$. Thus the functions could have real or complex values.

Definition 7.1.1 *Let f be a function which is defined on $D(f)$ where $D(f) \supseteq (x-r,x)\cup(x,x+r)$ for some $r>0$. Note that f is not necessarily defined at x. Then*

$$\lim_{y\to x} f(y) = L$$

if and only if the following condition holds. For all $\varepsilon>0$ there exists $\delta>0$ such that if

$$0<|y-x|<\delta,$$

then,

$$|L-f(y)|<\varepsilon.$$

If everything is the same as the above, except y is required to be larger than x and f is only required to be defined on $(x,x+r)$, then the notation is

$$\lim_{y\to x+} f(y) = L.$$

If f is only required to be defined on $(x-r,x)$ and y is required to be less than x,with the same conditions above, we write

$$\lim_{y\to x-} f(y) = L.$$

Limits are also taken as a variable "approaches" infinity. Of course nothing is "close" to infinity and so this requires a slightly different definition.

$$\lim_{x\to\infty} f(x) = L$$

if for every $\varepsilon>0$ there exists l such that whenever $x>l$,

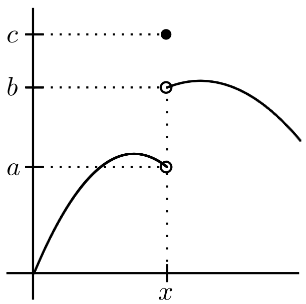
$$|f(x)-L|<\varepsilon \tag{7.1}$$

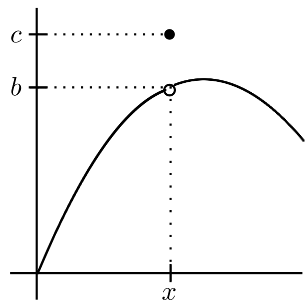
and

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\varepsilon > 0$ *there exists* l *such that whenever* $x < l$, *7.1 holds.*

The following pictures illustrate some of these definitions.





In the left picture is shown the graph of a function. Note the value of the function at x equals c while $\lim_{y \to x+} f(y) = b$ and $\lim_{y \to x-} f(y) = a$. In the second picture, $\lim_{y \to x} f(y) = b$. Note that the value of the function at the point x has nothing to do with the limit of the function in any of these cases. **The value of a function at x is irrelevant to the value of the limit at x!** This must always be kept in mind. You do not evaluate interesting limits by computing $f(x)$! In the above picture, $f(x)$ is always wrong! It may be the case that $f(x)$ is right but this is merely a happy coincidence when it occurs and as explained below in Theorem 7.1.6, this is sometimes equivalent to f being continuous at x.

Theorem 7.1.2 *If* $\lim_{y \to x} f(y) = L$ *and* $\lim_{y \to x} f(y) = L_1$, *then* $L = L_1$.

Proof: Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $0 < |y - x| < \delta$, then

$$|f(y) - L| < \varepsilon, \ |f(y) - L_1| < \varepsilon.$$

Therefore, for such y,

$$|L - L_1| \leq |L - f(y)| + |f(y) - L_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $L = L_1$.

The above theorem holds for any of the kinds of limits presented in the above definition.

Another concept is that of a function having either ∞ or $-\infty$ as a limit. In this case, the values of the function do not ever get close to their target because nothing can be close to $\pm\infty$. Roughly speaking, the limit of the function equals ∞ if the values of the function are ultimately larger than any given number. More precisely:

Definition 7.1.3 *If* $f(x) \in \mathbb{R}$, *then* $\lim_{y \to x} f(x) = \infty$ *if for every number* l, *there exists* $\delta > 0$ *such that whenever* $|y - x| < \delta$, *then* $f(x) > l$. $\lim_{x \to \infty} f(x) = \infty$ *if for all* k, *there exists* l *such that* $f(x) > k$ *whenever* $x > l$. *One sided limits and limits as the variable approaches* $-\infty$, *are defined similarly.*

It may seem there is a lot to memorize here. In fact, this is not so because all the definitions are intuitive when you understand them. Everything becomes much easier when you understand the definitions. This is usually the way it works in mathematics.

In the following theorem it is assumed the domains of the functions are such that the various limits make sense. Thus, if $\lim_{y \to x}$ is used, it is to be understood the function is defined on $(x-\delta,x)\cup(x,x+\delta)$ for some $\delta>0$. However, to avoid having to state things repetitively this symbol will be written to symbolize $\lim_{y\to x+}$ or $\lim_{y\to x-}$ and in either of these cases, it is understood the function is defined on an appropriate set so that the limits make sense. Thus in the case of $\lim_{y\to x+}$ the function is understood to be defined on an interval of the form $(x,x+\delta)$ with a similar convention holding for $\lim_{y\to x-}$.

Theorem 7.1.4 *In this theorem, the symbol* $\lim_{y\to x}$ *denotes any of the limits described above. Suppose* $\lim_{y\to x} f(y)=L$ *and* $\lim_{y\to x} g(y)=K$ *where* K *and* L *are numbers, not* $\pm\infty$. *Then if* a, b *are numbers,*

$$\lim_{y\to x}\left(af(y)+bg(y)\right)=aL+bK, \tag{7.2}$$

$$\lim_{y\to x} fg(y)=LK \tag{7.3}$$

and if $K\neq 0$,

$$\lim_{y\to x}\frac{f(y)}{g(y)}=\frac{L}{K}. \tag{7.4}$$

Also, if h *is a continuous function defined in some interval containing* L, *then*

$$\lim_{y\to x} h\circ f(y)=h(L). \tag{7.5}$$

Suppose f *is real valued and* $\lim_{y\to x} f(y)=L$. *If* $f(y)\leq a$ *all* y *near* x *either to the right or to the left of* x, *then* $L\leq a$ *and if* $f(y)\geq a$ *then* $L\geq a$.

Proof: The proof of 7.2 is left for you. It is like a corresponding theorem for continuous functions. Next consider 7.3. Let $\varepsilon>0$ be given. Then by the triangle inequality,

$$\begin{aligned}|fg(y)-LK| &\leq |fg(y)-f(y)K|+|f(y)K-LK|\\ &\leq |f(y)|\,|g(y)-K|+|K|\,|f(y)-L|.\end{aligned} \tag{7.6}$$

There exists δ_1 such that if $0<|y-x|<\delta_1$, then

$$|f(y)-L|<1,$$

and so for such y, and the triangle inequality, $|f(y)|<1+|L|$. Therefore, for $0<|y-x|<\delta_1$,

$$|fg(y)-LK|\leq(1+|K|+|L|)\left[|g(y)-K|+|f(y)-L|\right]. \tag{7.7}$$

Now let $0<\delta_2$ be such that for $0<|x-y|<\delta_2$,

$$|f(y)-L|<\frac{\varepsilon}{2(1+|K|+|L|)},\ |g(y)-K|<\frac{\varepsilon}{2(1+|K|+|L|)}.$$

Then letting $0<\delta\leq\min(\delta_1,\delta_2)$, it follows from 7.7 that

$$|fg(y)-LK|<\varepsilon$$

and this proves 7.3. Limits as $x\to\pm\infty$ and one sided limits are handled similarly.

The proof of 7.4 is left to you. It is just like the theorem about the quotient of continuous functions being continuous provided the function in the denominator is non zero at the point of interest.

Consider 7.5. Since h is continuous at L, it follows that for $\varepsilon>0$ given, there exists $\eta>0$ such that if $|y-L|<\eta$, then

$$|h(y)-h(L)|<\varepsilon$$

Now since $\lim_{y\to x} f(y) = L$, there exists $\delta > 0$ such that if $0 < |y-x| < \delta$, then

$$|f(y) - L| < \eta.$$

Therefore, if $0 < |y-x| < \delta$,

$$|h(f(y)) - h(L)| < \varepsilon.$$

The same theorem holds for one sided limits and limits as the variable moves toward $\pm\infty$. The proofs are left to you. They are minor modifications of the above.

It only remains to verify the last assertion. Assume $f(y) \leq a$. It is required to show that $L \leq a$. If this is not true, then $L > a$. Letting ε be small enough that $a < L - \varepsilon$, it follows that ultimately, for y close enough to x, $f(y) \in (L - \varepsilon, L + \varepsilon)$ which requires $f(y) > a$ contrary to assumption.

A very useful theorem for finding limits is called the squeezing theorem.

Theorem 7.1.5 *Suppose f, g, h are real valued functions and that*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x)$$

and for all x near a,

$$f(x) \leq h(x) \leq g(x).$$

Then

$$\lim_{x\to a} h(x) = L.$$

Proof: If $L \geq h(x)$, then

$$|h(x) - L| \leq |f(x) - L|.$$

If $L < h(x)$, then

$$|h(x) - L| \leq |g(x) - L|.$$

Therefore,

$$|h(x) - L| \leq |f(x) - L| + |g(x) - L|.$$

Now let $\varepsilon > 0$ be given. There exists δ_1 such that if $0 < |x - a| < \delta_1$,

$$|f(x) - L| < \varepsilon/2$$

and there exists δ_2 such that if $0 < |x - a| < \delta_2$, then

$$|g(x) - L| < \varepsilon/2.$$

Letting $0 < \delta \leq \min(\delta_1, \delta_2)$, if $0 < |x - a| < \delta$, then

$$\begin{aligned} |h(x) - L| &\leq |f(x) - L| + |g(x) - L| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves the theorem.

Theorem 7.1.6 *For $f : I \to \mathbb{R}$, and I is an interval of the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$, then f is continuous at $x \in I$ if and only if $\lim_{y\to x} f(y) = f(x)$.*

Proof: You fill in the details. Compare the definition of continuous and the definition of the limit just given.

Example 7.1.7 *Find* $\lim_{x\to 3}\frac{x^2-9}{x-3}$.

Note that $\frac{x^2-9}{x-3}=x+3$ whenever $x\neq 3$. Therefore, if $0<|x-3|<\varepsilon$,

$$\left|\frac{x^2-9}{x-3}-6\right|=|x+3-6|=|x-3|<\varepsilon.$$

It follows from the definition that this limit equals 6.

You should be careful to note that in the definition of limit, the variable **never equals the thing it is getting close to**. In this example, x is never equal to 3. This is very significant because, in interesting limits, the function whose limit is being taken will not be defined at the point of interest. The habit students acquire of plugging in the point to take the limit is only good on useless and uninteresting limits which are not good for anything other than to give a busy work exercise.

Example 7.1.8 *Let*

$$f(x)=\frac{x^2-9}{x-3}\ \textit{if } x\neq 3.$$

How should f be defined at $x=3$ so that the resulting function will be continuous there?

The limit of this function equals 6 because for $x\neq 3$,

$$\frac{x^2-9}{x-3}=\frac{(x-3)(x+3)}{x-3}=x+3$$

Therefore, by Theorem 7.1.6 it is necessary to define $f(3)\equiv 6$.

Example 7.1.9 *Find* $\lim_{x\to\infty}\frac{x}{1+x}$.

Write $\frac{x}{1+x}=\frac{1}{1+(1/x)}$. Now it seems clear that $\lim_{x\to\infty}1+(1/x)=1\neq 0$. Therefore, Theorem 7.1.4 implies

$$\lim_{x\to\infty}\frac{x}{1+x}=\lim_{x\to\infty}\frac{1}{1+(1/x)}=\frac{1}{1}=1.$$

Example 7.1.10 *Show* $\lim_{x\to a}\sqrt{x}=\sqrt{a}$ *whenever* $a\geq 0$. *In the case that* $a=0$, *take the limit from the right.*

There are two cases. First consider the case when $a>0$. Let $\varepsilon>0$ be given. Multiply and divide by $\sqrt{x}+\sqrt{a}$. This yields

$$\left|\sqrt{x}-\sqrt{a}\right|=\left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right|.$$

Now let $0<\delta_1<a/2$. Then if $|x-a|<\delta_1, x>a/2$ and so

$$\begin{aligned}\left|\sqrt{x}-\sqrt{a}\right|=\left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right|&\leq\frac{|x-a|}{\left(\sqrt{a}/\sqrt{2}\right)+\sqrt{a}}\\&\leq\frac{2\sqrt{2}}{\sqrt{a}}|x-a|.\end{aligned}$$

Now let $0<\delta\leq\min\left(\delta_1,\frac{\varepsilon\sqrt{a}}{2\sqrt{2}}\right)$. Then for $0<|x-a|<\delta$,

$$\left|\sqrt{x}-\sqrt{a}\right|\leq\frac{2\sqrt{2}}{\sqrt{a}}|x-a|<\frac{2\sqrt{2}}{\sqrt{a}}\frac{\varepsilon\sqrt{a}}{2\sqrt{2}}=\varepsilon.$$

Next consider the case where $a=0$. In this case, let $\varepsilon>0$ and let $\delta=\varepsilon^2$. Then if $0<x-0<\delta=\varepsilon^2$, it follows that $0\leq\sqrt{x}<\left(\varepsilon^2\right)^{1/2}=\varepsilon$.

7.2 Exercises

1. Find the following limits if possible

   (a) $\lim_{x\to 0+} \frac{|x|}{x}$

   (b) $\lim_{x\to 0+} \frac{x}{|x|}$

   (c) $\lim_{x\to 0-} \frac{|x|}{x}$

   (d) $\lim_{x\to 4} \frac{x^2-16}{x+4}$

   (e) $\lim_{x\to 3} \frac{x^2-9}{x+3}$

   (f) $\lim_{x\to -2} \frac{x^2-4}{x-2}$

   (g) $\lim_{x\to \infty} \frac{x}{1+x^2}$

   (h) $\lim_{x\to \infty} -2\frac{x}{1+x^2}$

2. Find $\lim_{h\to 0} \frac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h}$.

3. Find $\lim_{x\to 4} \frac{\sqrt[4]{x}-\sqrt{2}}{\sqrt{x}-2}$.

4. Find $\lim_{x\to \infty} \frac{\sqrt[5]{3x}+\sqrt[4]{x}+7\sqrt{x}}{\sqrt{3x+1}}$.

5. Find $\lim_{x\to \infty} \frac{(x-3)^{20}(2x+1)^{30}}{(2x^2+7)^{25}}$.

6. Find $\lim_{x\to 2} \frac{x^2-4}{x^3+3x^2-9x-2}$.

7. Find $\lim_{x\to \infty} \left(\sqrt{1-7x+x^2}-\sqrt{1+7x+x^2}\right)$.

8. Prove Theorem 7.1.2 for right, left and limits as $y\to\infty$.

9. Prove from the definition that $\lim_{x\to a} \sqrt[3]{x} = \sqrt[3]{a}$ for all $a\in\mathbb{R}$. **Hint:** You might want to use the formula for the difference of two cubes,

$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right).$$

10. Find $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$.

11. Prove Theorem 7.1.6 from the definitions of limit and continuity.

12. Find $\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$

13. Find $\lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

14. Find $\lim_{x\to -3} \frac{x^3+27}{x+3}$

15. Find $\lim_{h\to 0} \frac{\sqrt{(3+h)^2}-3}{h}$ if it exists.

16. Find the values of x for which $\lim_{h\to 0} \frac{\sqrt{(x+h)^2}-x}{h}$ exists and find the limit.

17. Find $\lim_{h\to 0} \frac{\sqrt[3]{(x+h)}-\sqrt[3]{x}}{h}$ if it exists. Here $x\neq 0$.

18. Suppose $\lim_{y\to x+} f(y) = L_1 \neq L_2 = \lim_{y\to x-} f(y)$. Show $\lim_{y\to x} f(x)$ does not exist. **Hint:** Roughly, the argument goes as follows: For $|y_1 - x|$ small and $y_1 > x$, $|f(y_1) - L_1|$ is small. Also, for $|y_2 - x|$ small and $y_2 < x$, $|f(y_2) - L_2|$ is small. However, if a limit existed, then $f(y_2)$ and $f(y_1)$ would both need to be close to some number and so both L_1 and L_2 would need to be close to some number. However, this is impossible because they are different.

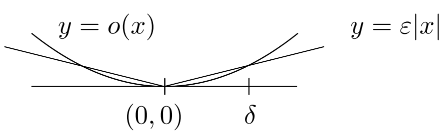
19. Let $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$. Find $\lim_{x\to 0}(\lim_{y\to 0} f(x, y))$ and $\lim_{y\to 0}(\lim_{x\to 0} f(x, y))$. If you did it right you got -1 for one answer and 1 for the other. What does this tell you about interchanging limits?

20. The whole presentation of limits above is too specialized. Let D be the domain of a function, f. A point x not necessarily in D, is said to be a limit point of D if $B(x, r)$ contains a point of D not equal to x for every $r > 0$. Now define the concept of limit in the same way as above and show that the limit is well defined if it exists. That is, if x is a limit point of D and $\lim_{y\to x} f(x) = L_1$ and $\lim_{y\to x} f(x) = L_2$, then $L_1 = L_2$. Is it possible to take a limit of a function at a point not a limit point of D? What would happen to the above property of the limit being well defined? Is it reasonable to define continuity at isolated points, those points which are not limit points, in terms of a limit as is often done in calculus books?

21. If f is an increasing function which is bounded above by a constant, M, show that $\lim_{x\to\infty} f(x)$ exists. Give a similar theorem for decreasing functions.

7.3 The Definition Of The Derivative

The following picture of a function, $y = o(x)$ is an example of one which appears to be tangent to the line $y = 0$ at the point $(0, 0)$.



You see in this picture, the graph of the function $y = \varepsilon|x|$ also where $\varepsilon > 0$ is just a positive number. Note there exists $\delta > 0$ such that if $|x| < \delta$, then $|o(x)| < \varepsilon|x|$ or in other words,

$$\frac{|o(x)|}{|x|} < \varepsilon.$$

You might draw a few other pictures of functions which would have the appearance of being tangent to the line $y = 0$ at the point $(0, 0)$ and observe that in every case, it will follow that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x| < \delta$, then

$$\frac{|o(x)|}{|x|} < \varepsilon. \tag{7.8}$$

In other words, a reasonable way to say a function is tangent to the line $y = 0$ at $(0, 0)$ is to say for all $\varepsilon > 0$ there exists $\delta > 0$ such that 7.8 holds. In other words, the function

$y = o(x)$ is tangent at $(0,0)$ if and only if

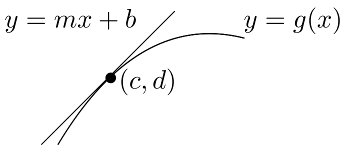
$$\lim_{x\to 0}\frac{|o(x)|}{|x|}=0.$$

Definition 7.3.1 *A function $y = k(x)$ is said to be $o(x)$ if*

$$\lim_{x\to 0}\frac{|k(x)|}{|x|}=0 \tag{7.9}$$

As was just discussed, in the case where $x \in \mathbb{R}$ and k is a function having values in $\mathbb{R}$ this is geometrically the same as saying the function is tangent to the line $y = 0$ at the point $(0,0)$. This terminology is used like an adjective. $k(x)$ is $o(x)$ means 7.9 holds. Thus $o(x) = 5o(x)$, $o(x)+o(x) = o(x)$, etc. The usage is very imprecise and sloppy, leaving out exactly the details which are of absolutely no significance in what is about to be discussed. It is this sloppiness which makes the notation so useful. It prevents you from fussing with things which do not matter.

Now consider the case of the function, $y = g(x)$ tangent to $y = b + mx$ at the point (c,d).



Thus, in particular, $g(c) = b + mc = d$. Then letting $x = c + h$, it follows x is close to c if and only if h is close to 0. Consider then the two functions

$$y = g(c+h),\ y = b + m(c+h).$$

If they are tangent as shown in the above picture, you should have the function

$$\begin{aligned} k(h) &\equiv g(c+h) - (b + m(c+h)) \\ &= g(c+h) - (b+mc) - mh \\ &= g(c+h) - g(c) - mh \end{aligned}$$

tangent to $y = 0$ at the point $(0,0)$. As explained above, the precise meaning of this function being tangent as described is to have $k(h) = o(h)$. This motivates (I hope) the following definition of the derivative which is the precise definition free of pictures and heuristics.

Definition 7.3.2 *Let g be a $\mathbb{F}$ valued function defined on an open set in $\mathbb{F}$ containing c. Then $g'(c)$ is the number, if it exists, which satisfies*

$$g(c+h) - g(c) - g'(c)h = o(h)$$

where $o(h)$ is defined in Definition 7.3.1.

The above definition is more general than what will be extensively discussed here. I will usually consider the case where the function is defined on some interval contained in $\mathbb{R}$. In this context, the definition of derivative can also be extended to include right and left derivatives.

Definition 7.3.3 *Let g be a function defined on an interval, $[c,b)$. Then $g'_+(c)$ is the number, if it exists, which satisfies*

$$g_+(c+h) - g_+(c) - g'_+(c)\,h = o(h)$$

where $o(h)$ is defined in Definition 7.3.1 except you only consider positive h. Thus

$$\lim_{h\to 0+} \frac{|o(h)|}{|h|} = 0.$$

This is the derivative from the right. Let g be a function defined on an interval, $(a,c]$. Then $g'_-(c)$ is the number, if it exists, which satisfies

$$g_-(c+h) - g_-(c) - g'_-(c)\,h = o(h)$$

where $o(h)$ is defined in Definition 7.3.1 except you only consider negative h. Thus

$$\lim_{h\to 0-} \frac{|o(h)|}{|h|} = 0.$$

This is the derivative from the left.

I will not pay any attention to these distinctions from now on. In particular I will not write g'_- and g'_+ unless it is necessary. If the domain of a function defined on a subset of $\mathbb{R}$ is not open, it will be understood that at an endpoint, the derivative meant will be the appropriate derivative from the right or the left. First I need to show this is well defined because there cannot be two values for $g'(c)$.

Theorem 7.3.4 *The derivative is well defined because if*

$$\begin{aligned} g(c+h) - g(c) - m_1 h &= o(h) \\ g(c+h) - g(c) - m_2 h &= o(h) \end{aligned} \tag{7.10}$$

then $m_1 = m_2$.

Proof: Suppose 7.10. Then subtracting these,

$$(m_2 - m_1)\,h = o(h) - o(h) = o(h)$$

and so dividing by $h \neq 0$ and then taking a limit as $h \to 0$ gives

$$m_2 - m_1 = \lim_{h\to 0} \frac{o(h)}{h} = 0.$$

Note the same argument holds for derivatives from the right or the left also. This proves the theorem.

Now the derivative has been defined, here are some properties.

Lemma 7.3.5 *Suppose $g'(c)$ exists. Then there exists $\delta > 0$ such that if $|h| < \delta$,*

$$|g(c+h) - g(c)| < (|g'(c)| + 1)\,|h| \tag{7.11}$$

$$o(|g(c+h) - g(c)|) = o(h) \tag{7.12}$$

g is continuous at c.

Proof: This follows from the definition of $g'(c)$.

$$g(c+h) - g(c) - g'(c)h = o(h)$$

and so there exists $\delta > 0$ such that if $0 < |h| < \delta$,

$$\frac{|g(c+h) - g(c) - g'(c)h|}{|h|} < 1$$

By the triangle inequality,

$$|g(c+h) - g(c)| - |g'(c)h| \leq |g(c+h) - g(c) - g'(c)h| < |h|$$

and so

$$|g(c+h) - g(c)| < (|g'(c)| + 1)|h|$$

Next consider the second claim. By definition of the little o notation, there exists a $\delta_1 > 0$ such that if

$$|g(c+h) - g(c)| < \delta_1,$$

then

$$o(|g(c+h) - g(c)|) < \frac{\varepsilon}{|g'(c)| + 1}|g(c+h) - g(c)|. \tag{7.13}$$

But from the first inequality, if $|h| < \delta$, then

$$|g(c+h) - g(c)| < (|g'(c)| + 1)|h|$$

and so for $|h| < \min\left(\delta, \frac{\delta_1}{(|g'(c)|+1)}\right)$, it follows

$$|g(c+h) - g(c)| < (|g'(c)| + 1)|h| < \delta_1$$

and so from 7.13,

$$\begin{aligned} o(|g(c+h) - g(c)|) &< \frac{\varepsilon}{|g'(c)| + 1}|g(c+h) - g(c)| \\ &< \frac{\varepsilon}{|g'(c)| + 1}(|g'(c)| + 1)|h| = \varepsilon|h| \end{aligned}$$

and this shows

$$\lim_{h \to 0} \frac{o(|g(c+h) - g(c)|)}{|h|} = 0$$

because for nonzero h small enough,

$$\frac{o(|g(c+h) - g(c)|)}{|h|} < \varepsilon.$$

This proves 7.12.

The assertion about continuity follows right away from 7.11. Just let $h = x - c$ and the formula gives

$$|g(x) - g(c)| < (|g'(c)| + 1)|x - c|$$

This proves the theorem.

Of course some functions do not have derivatives at some points.

Example 7.3.6 *Let $f(x) = |x|$. Show $f'(0)$ does not exist.*

If it did exist, then

$$|h| - f'(0)\, h = o(h)$$

Hence, replacing h with $-h$,

$$|-h| - f'(0)(-h) = o(-h)$$

and so, subtracting these,

$$2f'(0)\, h = o(-h) - o(h) = o(h)$$

and so

$$2f'(0) = \frac{o(h)}{h}.$$

Now letting $h \to 0$, it follows $f'(0) = 0$ if it exists. However, this would say

$$|h| = o(h)$$

which is false. Thus $f'(0)$ cannot exist. However, this function has right derivatives at every point and also left derivatives at every point. For example, consider $f'(0)$ as a right derivative. For $h > 0$

$$f(h) - 0 - 1h = 0 = o(h)$$

and so $f'_+(0) = 1$. For $h < 0$,

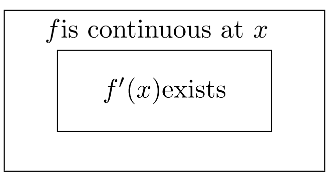
$$f(h) = -h$$

and so

$$f(h) - f(0) - (-1)\, h = 0 = o(h)$$

and so $f'_-(0) = -1$. You should show $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$.

The following diagram shows how continuity at a point and differentiability there are related.



7.4 Continuous And Nowhere Differentiable

How bad can it get in terms of a continuous function not having a derivative at some points? It turns out it can be the case the function is nowhere differentiable but everywhere continuous. An example of such a pathological function different than the one I am about to present was discovered by Weierstrass in 1872. Before showing this, here is a simple observation.

Lemma 7.4.1 *Suppose $f'(x)$ exists and let c be a number. Then letting $g(x) \equiv f(cx)$,*

$$g'(x) = cf'(cx).$$

Here the derivative refers to either the derivative, the left derivative, or the right derivative. Also, if $f(x) = a + bx$, then

$$f'(x) = b$$

where again, f' refers to either the left derivative, right derivative or derivative. Furthermore, in the case where $f(x) = a + bx$,

$$f(x+h) - f(x) = bh.$$

Proof: It is known

$$f(x+h) - f(x) - f'(x)h = o(h)$$

Therefore,

$$g(x+h) - g(x) = f(c(x+h)) - f(cx) = f'(cx)ch + o(ch)$$

and so

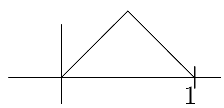
$$g(x+h) - g(x) - cf'(cx)h = o(ch) = o(h)$$

and so this proves the first part of the lemma. Now consider the last claim.

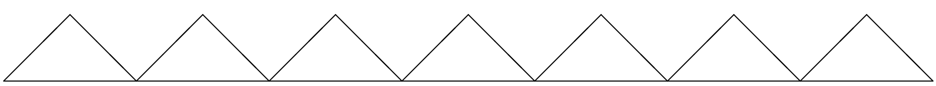
$$\begin{aligned} f(x+h) - f(x) &= a + b(x+h) - (a+bx) = bh \\ &= bh + 0 = bh + o(h). \end{aligned}$$

Thus $f'(x) = b$. This proves the lemma.

Now consider the following description of a function. The following is the graph of the function on $[0,1]$.



The height of the function is $1/2$ and the slope of the rising line is 1 while the slope of the falling line is -1. Now extend this function to the whole real line to make it periodic of period 1. This means $f(x+n) = f(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, the integers. In other words to find the graph of f on $[1,2]$ you simply slide the graph of f on $[0,1]$ a distance of 1 to get the same tent shaped thing on $[1,2]$. Continue this way. The following picture illustrates what a piece of the graph of this function looks like. Some might call it an infinite sawtooth.



Now define

$$g(x) \equiv \sum_{k=0}^{\infty} 4^{-k} f\left(4^k x\right).$$

Letting $M_k = 4^{-k}$, an application of the Weierstrass M test shows g is everywhere continuous. This is because each function in the sum is continuous and the series converges uniformly on $\mathbb{R}$.

Also note that f has a right derivative and a left derivative at every point, denoted by $f'(x)$ to save on notation, and $f'(x)$ equals either 1 or -1. Suppose $g'(x)$ exists at some point and let $h_m = b_m 4^{-m}$ where b_m equals either 1 or -1 chosen such that $4^{m-1}x$ and $4^{m-1}(x+b_m 4^{-m})$ are both in either $[N, N+\frac{1}{2})$ or $[N+\frac{1}{2}, N)$ for some integer N. This is certainly possible to do because for some N, $4^{m-1}x$ is in an interval of one of the two forms just described and these intervals are of length $1/2$ while the distance between those two points is $1/4$. Now consider what happens to $4^k x$ and $4^k(x+b_m 4^{-m})$ for $k < m-1$. If for some integer $M, 4^k(x+b_m 4^{-m})$ and $4^k x$ are on opposite sides of a number of the form $M+\frac{1}{2}$, say

$$4^k x < M + \frac{1}{2} < 4^k\left(x + b_m 4^{-m}\right),$$

then multiplying by 4^{m-1-k},

$$4^{m-1}x < \overbrace{4^{m-1-k}\left(M+\frac{1}{2}\right)}^{\text{integer}} < 4^{m-1}\left(x+b_m 4^{-m}\right)$$

which would require $4^{m-1}(x+b_m 4^{-m})$ and $4^{m-1}x$ to be on different sides of an integer contrary to the way b_m was chosen. Thus for each $k \leq m-1$, it follows from Lemma 7.4.1

$$f\left(4^k(x+h_m)\right) - f\left(4^k x\right) = f'\left(4^k x\right)4^k h_m \tag{7.14}$$

where $f'(4^k x)$ is either the right or the left derivative, depending on whether b_m is 1 or -1. If $b_m = 1$, it is the right derivative and if $b_m = -1$, it is the left derivative. Now

$$\begin{aligned} g(x+h_m) - g(x) &= g'(x)h_m + o(h_m) \\ &= \sum_{k=0}^{\infty} 4^{-k}\left(f\left(4^k(x+h_m)\right) - f\left(4^k x\right)\right) \\ &= \sum_{k=0}^{m-1} 4^{-k}\left(f\left(4^k(x+h_m)\right) - f\left(4^k x\right)\right) \end{aligned} \tag{7.15}$$

where the sum reduces to a finite sum because if $k > m-1$,

$$\begin{aligned} f\left(4^k(x+h_m)\right) - f\left(4^k x\right) &= f\left(4^k\left(x+\frac{\pm 1}{4^m}\right)\right) - f\left(4^k x\right) \\ &= f\left(4^k x \pm 4^{k-m}\right) - f\left(4^k x\right) \end{aligned}$$

since 4^{k-m} is an integer and f is periodic of period 1 which implies,

$$f\left(4^k(x+h_m)\right) = f\left(4^k x\right).$$

Now consider the sum in 7.15. From 7.14, this equals

$$\sum_{k=0}^{m-1} 4^{-k}\left(f'\left(4^k x\right)4^k h_m\right) = \sum_{k=0}^{m-1} f'\left(4^k x\right)h_m$$

where the derivative equals either a right or a left derivative. This is not important, only that $f'(4^k x) = \pm 1$. Now from 7.15 and dividing by h_m,

$$g'(x) + \frac{o(h_m)}{h_m} = \sum_{k=0}^{m-1} f'\left(4^k x\right)$$

and now this yields a contradiction because there exists a limit on the left of the equality as $m \to \infty$ but there can be no limit on the right because the k^{th} term of the series fails to converge to 0 as $k \to \infty$. This proves the following theorem.

Theorem 7.4.2 *There exists a function defined on $\mathbb{R}$ which is continuous and bounded but fails to have a derivative at any point.*

7.5 Finding The Derivative

Obviously there need to be simple ways of finding the derivative when it exists. There are rules of derivatives which make finding the derivative very easy. In the following theorem, the derivative could refer to right or left derivatives as well as regular derivatives.

Theorem 7.5.1 *Let a, b be numbers and suppose $f'(t)$ and $g'(t)$ exist. Then the following formulas are obtained.*

$$(af+bg)'(t) = af'(t) + bg'(t). \tag{7.16}$$

$$(fg)'(t) = f'(t)g(t) + f(t)g'(t). \tag{7.17}$$

The formula, 7.17 is referred to as the product rule.

If $f'(g(t))$ exists and $g'(t)$ exists, then $(f\circ g)'(t)$ also exists and

$$(f\circ g)'(t) = f'(g(t))g'(t).$$

This is called the chain rule. In this rule, for the sake of simiplicity, assume the derivatives are real derivatives, not derivatives from the right or the left. If $f(t) = t^n$ where n is any integer, then

$$f'(t) = nt^{n-1}. \tag{7.18}$$

Also, whenever $f'(t)$ exists,

$$f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

where this definition can be adjusted in the case where $t\in\mathbb{R}$ if the derivative is a right or left derivative by letting $h>0$ or $h<0$ only and considering a one sided limit. This is equivalent to

$$f'(t) = \lim_{s\to t}\frac{f(s)-f(t)}{t-s}$$

with the limit being one sided in the case of a left or right derivative.

Proof: 7.16 is left for you. Consider 7.17

$$fg(t+h) - fg(t) = f(t+h)g(t+h) - f(t)g(t+h) + f(t)g(t+h) - f(t)g(t)$$

$$= g(t+h)(f(t+h)-f(t)) + f(t)(g(t+h)-g(t))$$

$$\begin{aligned} &= g(t+h)(f'(t)h+o(h)) + f(t)(g'(t)+o(h)) \\ &= g(t)f'(t)h + f(t)g'(t) + f(t)o(h) \\ &\quad + (g(t+h)-g(t))f'(t)h + g(t+h)o(h) \end{aligned}$$

$$= g(t)f'(t)h + f(t)g'(t) + o(h)$$

because by Lemma 7.3.5, g is continuous at t and so

$$(g(t+h)-g(t))f'(t)h = o(h)$$

While $f(t)o(h)$ and $g(t+h)o(h)$ are both $o(h)$. This proves 7.17.

Next consider the chain rule. By Lemma 7.3.5 again,

$$f\circ g(t+h) = f(g(t+h)) - f(g(t))$$

$$= f(g(t) + (g(t+h) - g(t))) - f(g(t))$$

$$= f'(g(t))(g(t+h) - g(t)) + o((g(t+h) - g(t)))$$

$$= f'(g(t))(g(t+h) - g(t)) + o(h)$$

$$\begin{aligned} &= f'(g(t))(g'(t) + o(h)) + o(h) \\ &= f'(g(t))g'(t) + o(h). \end{aligned}$$

This proves the chain rule.

Now consider the claim about $f(t) = t^n$ for n an integer. If $n = 0, 1$ the desired conclusion follows from Lemma 7.4.1. Suppose the claim is true for $n \geq 1$. Then let $f_{n+1}(t) = t^{n+1} = f_n(t)t$ where $f_n(t) \equiv t^n$. Then by the product rule, induction and the validity of the assertion for $n = 1$,

$$f'_{n+1}(t) = f'_n(t)t + f_n(t) = tnt^{n-1} + t^n = nt^{n+1}$$

and so the assertion is proved for all $n \geq 0$. Consider now $n = -1$.

$$\begin{aligned} (t+h)^{-1} - t^{-1} &= \frac{-1}{t(t+h)}h = \frac{-1}{t^2}h + \left(\frac{-1}{t(t+h)} + \frac{1}{t^2}\right)h \\ &= \frac{-1}{t^2}h + \frac{h^2}{t^2(t+h)} = -\frac{1}{t^2}h + o(h) = (-1)t^{-2}h + o(h) \end{aligned}$$

Therefore, the assertion is true for $n = -1$. Now consider $f(t) = t^{-n}$ where n is a positive integer. Then $f(t) = (t^n)^{-1}$ and so by the chain rule,

$$f'(t) = (-1)(t^n)^{-2} nt^{n-1} = -nt^{-n-1}.$$

This proves 7.18.

Finally, if $f'(t)$ exists,

$$f'(t)h + o(h) = f(t+h) - f(t).$$

Divide by h and take the limit as $h \to 0$, either a regular limit of in the special case where $t \in \mathbb{R}$ from the right or from the left. Then this yields

$$f'(t) = \lim_{h\to 0}\left(\frac{f(t+h) - f(t)}{h} + \frac{o(h)}{h}\right) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{h}.$$

This proves the theorem.

Note the last part is the usual definition of the derivative given in beginning calculus courses. There is nothing wrong with doing it this way from the beginning for a function of only one variable but it is not the right way to think of the derivative and does not generalize to the case of functions of many variables where the definition given in terms of $o(h)$ does.

Corollary 7.5.2 *Let $f'(t), g'(t)$ both exist and $g(t) \neq 0$, then the quotient rule holds.*

$$\left(\frac{f}{g}\right)' = \frac{f'(t)g(t) - f(t)g'(t)}{g(t)^2}$$

Proof: This is left to you. Use the chain rule and the product rule.

Higher order derivatives are defined in the usual way.

$$f'' \equiv (f')'$$

etc. Also the Leibniz notation is defined by

$$\frac{dy}{dx} = f'(x) \text{ where } y = f(x)$$

and the second derivative is denoted as

$$\frac{d^2y}{dx^2}$$

with various other higher order derivatives defined in the usual way.

The chain rule has a particularly attractive form in Leibniz's notation. Suppose $y = g(u)$ and $u = f(x)$. Thus $y = g \circ f(x)$. Then from the above theorem

$$\begin{aligned}(g \circ f)'(x) &= g'(f(x)) f'(x) \\ &= g'(u) f'(x)\end{aligned}$$

or in other words,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

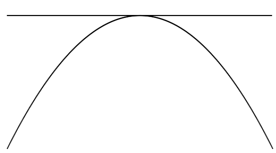
Notice how the du cancels. This particular form is a very useful crutch and is used extensively in applications.

7.6 Mean Value Theorem And Local Extreme Points

When you are on top of a hill, you are at a local maximum although there may be other hills higher than the one on which you are standing. Similarly, when you are at the bottom of a valley, you are at a local minimum even though there may be other valleys deeper than the one you are in. The word, "local" is applied to the situation because if you confine your attention only to points close to your location, you are indeed at either the top or the bottom.

Definition 7.6.1 *Let $f : D(f) \to \mathbb{R}$ where here $D(f)$ is only assumed to be some subset of $\mathbb{F}$. Then $x \in D(f)$ is a local minimum (maximum) if there exists $\delta > 0$ such that whenever $y \in B(x, \delta) \cap D(f)$, it follows $f(y) \geq (\leq) f(x)$. The plural of minimum is minima and the plural of maximum is maxima.*

Derivatives can be used to locate local maxima and local minima. The following picture suggests how to do this. This picture is of the graph of a function having a local maximum and the tangent line to it.



Note how the tangent line is horizontal. If you were not at a local maximum or local minimum, the function would be falling or climbing and the tangent line would not be horizontal.

Theorem 7.6.2 *Suppose $f : U \to \mathbb{R}$ where U is an open subset of $\mathbb{F}$ and suppose $x \in U$ is a local maximum or minimum. Then $f'(x) = 0$.*

Proof: Suppose x is a local maximum and let $\delta > 0$ is so small that $B(x, \delta) \subseteq U$. Then for $|h| < \delta$, both x and $x + h$ are contained in $B(x, \delta) \subseteq U$. Then letting h be real and positive,

$$f'(x) h + o(h) = f(x + h) - f(x) \leq 0.$$

Then dividing by h it follows from Theorem 7.1.4 on Page 115,

$$f'(x) = \lim_{h \to 0} \left(f'(x) + \frac{o(h)}{h} \right) \leq 0$$

Next let $|h| < \delta$ and h is real and negative. Then

$$f'(x) h + o(h) = f(x + h) - f(x) \leq 0.$$

Then dividing by h

$$f'(x) = \lim_{h \to 0} f'(x) + \frac{o(h)}{h} \geq 0$$

Thus $f'(x) = 0$. The case where x is a local minimum is handled similarly. Alternatively, you could apply what was just shown to $-f(x)$. This proves the theorem.[1]

Points at which the derivative of a function equals 0 are sometimes called critical points. Included in the set of critical points are those points where f' fails to exist.

7.7 Exercises

1. If $f'(x) = 0$, is it necessary that x is either a local minimum or local maximum? **Hint:** Consider $f(x) = x^3$.

2. A continuous function, f defined on $[a, b]$ is to be maximized. It was shown above in Theorem 7.6.2 that if the maximum value of f occurs at $x \in (a, b)$, and if f is differentiable there, then $f'(x) = 0$. However, this theorem does not say anything about the case where the maximum of f occurs at either a or b. Describe how to find the point of $[a, b]$ where f achieves its maximum. Does f have a maximum? Explain.

3. Show that if the maximum value of a function f differentiable on $[a, b]$ occurs at the right endpoint, then for all $h > 0, f'(b) h \geq 0$. This is an example of a variational inequality. Describe what happens if the maximum occurs at the left end point and give a similar variational inequality. What is the situation for minima?

4. Find the maximum and minimum values and the values of x where these are achieved for the function, $f(x) = x + \sqrt{25 - x^2}$.

[1] Actually, the case where the function is defined on an open subset of $\mathbb{F}$ and yet has real values is not too interesting. However, this is information which depends on the theory of functions of a complex variable which is not being considered in this book.

5. A piece of wire of length L is to be cut in two pieces. One piece is bent into the shape of an equilateral triangle and the other piece is bent to form a square. How should the wire be cut to maximize the sum of the areas of the two shapes? How should the wire be bent to minimize the sum of the areas of the two shapes? **Hint:** Be sure to consider the case where all the wire is devoted to one of the shapes separately. This is a possible solution even though the derivative is not zero there.

6. Lets find the point on the graph of $y = \frac{x^2}{4}$ which is closest to $(0, 1)$. One way to do it is to observe that a typical point on the graph is of the form $\left(x, \frac{x^2}{4}\right)$ and then to minimize the function, $f(x) = x^2 + \left(\frac{x^2}{4} - 1\right)^2$. Taking the derivative of f yields $x + \frac{1}{4}x^3$ and setting this equal to 0 leads to the solution, $x = 0$. Therefore, the point closest to $(0, 1)$ is $(0, 0)$. Now lets do it another way. Lets use $y = \frac{x^2}{4}$ to write $x^2 = 4y$. Now for (x, y) on the graph, it follows it is of the form $\left(\sqrt{4y}, y\right)$. Therefore, minimize $f(y) = 4y + (y - 1)^2$. Take the derivative to obtain $2 + 2y$ which requires $y = -1$. However, on this graph, y is never negative. What on earth is the problem?

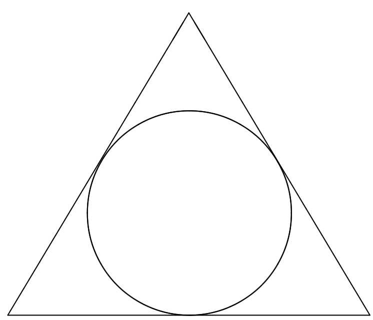
7. Find the dimensions of the largest rectangle that can be inscribed in the ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

8. A function, f, is said to be odd if $f(-x) = f(x)$ and a function is said to be even if $f(-x) = f(x)$. Show that if f' is even, then f is odd and if f' is odd, then f is even. Sketch the graph of a typical odd function and a typical even function.

9. Find the point on the curve, $y = \sqrt{25 - 2x}$ which is closest to $(0, 0)$.

10. A street is 200 feet long and there are two lights located at the ends of the street. One of the lights is $\frac{1}{8}$ times as bright as the other. Assuming the brightness of light from one of these street lights is proportional to the brightness of the light and the reciprocal of the square of the distance from the light, locate the darkest point on the street.

11. Find the volume of the smallest right circular cone which can be circumscribed about a sphere of radius 4 inches.



12. Show that for r a rational number and $y = x^r$, it must be the case that if this function is differentiable, then $y' = rx^{r-1}$.

13. Let f be a continuous function defined on $[a, b]$. Let $\varepsilon > 0$ be given. Show there exists a polynomial, p such that for all $x \in [a, b]$,

$$|f(x) - p(x)| < \varepsilon.$$

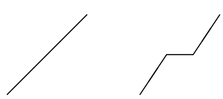
This follows from the Weierstrass approximation theorem, Theorem 6.9.3. Now here is the interesting part. Show there exists a function, g which is also continuous on $[a,b]$ and for all $x \in [a,b]$,

$$|f(x) - g(x)| < \varepsilon$$

but g has no derivative at any point. Thus there are enough nowhere differentiable functions that any continuous function is uniformly close to one. Explain why every continuous function is the uniform limit of nowhere differentiable functions. Also explain why every nowhere differentiable continuous function is the uniform limit of polynomials. **Hint:** You should look at the construction of the nowhere differentiable function which is everywhere continuous, given above.

14. Consider the following nested sequence of compact sets, $\{P_n\}$. Let $P_1 = [0,1]$, $P_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, etc. To go from P_n to P_{n+1}, delete the open interval which is the middle third of each closed interval in P_n. Let $P = \cap_{n=1}^{\infty} P_n$. By Problem 13 on Page 58, $P \neq \emptyset$. If you have not worked this exercise, now is the time to do it. Show the total length of intervals removed from $[0,1]$ is equal to 1. If you feel ambitious also show there is a one to one onto mapping of $[0,1]$ to P. The set P is called the Cantor set. Thus P has the same number of points in it as $[0,1]$ in the sense that there is a one to one and onto mapping from one to the other even though the length of the intervals removed equals 1. **Hint:** There are various ways of doing this last part but the most enlightenment is obtained by exploiting the construction of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the theorems in the chapter on set theory related to the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0,1]$. If you do this right it will provide a construction which is very useful to prove some even more surprising theorems which you may encounter later if you study compact metric spaces.

15. $\uparrow$ Consider the sequence of functions defined in the following way. Let $f_1(x) = x$ on $[0,1]$. To get from f_n to f_{n+1}, let $f_{n+1} = f_n$ on all intervals where f_n is constant. If f_n is nonconstant on $[a,b]$, let $f_{n+1}(a) = f_n(a)$, $f_{n+1}(b) = f_n(b)$, f_{n+1} is piecewise linear and equal to $\frac{1}{2}(f_n(a) + f_n(b))$ on the middle third of $[a,b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.



Show $\{f_n\}$ converges uniformly on $[0,1]$. If $f(x) = \lim_{n\to\infty} f_n(x)$, show that $f(0) = 0$, $f(1) = 1$, f is continuous, and $f'(x) = 0$ for all $x \notin P$ where P is the Cantor set of Problem 14. This function is called the Cantor function.It is a very important example to remember especially for those who like mathematical pathology. Note it has derivative equal to zero on all those intervals which were removed and whose total length was equal to 1 and yet it succeeds in climbing from 0 to 1. Isn't this amazing? **Hint:** This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.

16. Let

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Now let $g(x) = x^2 f(x)$. Find where g is continuous and differentiable if anywhere.

7.8 Mean Value Theorem

The mean value theorem is possibly the most important theorem about the derivative of a function of one variable. It pertains only to a real valued function of a real variable. The best versions of many other theorems depend on this fundamental result. The mean value theorem is based on the following special case known as Rolle's theorem[2]. It is an existence theorem and like the other existence theorems in analysis, it depends on the completeness axiom.

Theorem 7.8.1 *Suppose $f : [a, b] \to \mathbb{R}$ is continuous,*

$$f(a) = f(b),$$

and

$$f : (a, b) \to \mathbb{R}$$

has a derivative at every point of (a, b). Then there exists $x \in (a, b)$ such that $f'(x) = 0$.

Proof: Suppose first that $f(x) = f(a)$ for all $x \in [a, b]$. Then any $x \in (a, b)$ is a point such that $f'(x) = 0$. If f is not constant, either there exists $y \in (a, b)$ such that $f(y) > f(a)$ or there exists $y \in (a, b)$ such that $f(y) < f(b)$. In the first case, the maximum of f is achieved at some $x \in (a, b)$ and in the second case, the minimum of f is achieved at some $x \in (a, b)$. Either way, Theorem 7.6.2 implies $f'(x) = 0$. This proves Rolle's theorem.

The next theorem is known as the Cauchy mean value theorem. It is the best version of this important theorem.

Theorem 7.8.2 *Suppose f, g are continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x \in (a, b)$ such that*

$$f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a)).$$

Proof: Let

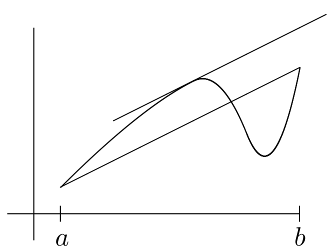
$$h(x) \equiv f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Then letting $x = a$ and then letting $x = b$, a short computation shows $h(a) = h(b)$. Also, h is continuous on $[a, b]$ and differentiable on (a, b). Therefore Rolle's theorem applies and there exists $x \in (a, b)$ such that

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)) = 0.$$

This proves the theorem.

Letting $g(x) = x$, the usual version of the mean value theorem is obtained. Here is the usual picture which describes the theorem.



[2]Rolle is remembered for Rolle's theorem and not for anything else he did. Ironically, he did not like calculus.

Corollary 7.8.3 *Let f be a continuous real valued function defined on $[a,b]$ and differentiable on (a,b). Then there exists $x \in (a,b)$ such that $f(b) - f(a) = f'(x)(b-a)$.*

Corollary 7.8.4 *Suppose $f'(x) = 0$ for all $x \in (a,b)$ where $a \geq -\infty$ and $b \leq \infty$. Then $f(x) = f(y)$ for all $x, y \in (a,b)$. Thus f is a constant.*

Proof: If this is not true, there exists x_1 and x_2 such that $f(x_1) \neq f(x_2)$. Then by the mean value theorem,

$$0 \neq \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(z)$$

for some z between x_1 and x_2. This contradicts the hypothesis that $f'(x) = 0$ for all x. This proves the theorem in the case that f has real values. In the general case,

$$f(x+h) - f(x) - 0h = o(h).$$

Then taking the real part of both sides,

$$\operatorname{Re} f(x+h) - \operatorname{Re} f(x) = \operatorname{Re} o(h) = o(h)$$

and so $\operatorname{Re} f'(x) = 0$ and by the first part, $\operatorname{Re} f$ must be a constant. The same reasoning applies to $\operatorname{Im} f$ and this proves the corollary.

Corollary 7.8.5 *Suppose $f : \mathbb{C} \to \mathbb{C}$ and $f'(x) = 0$ for all x. Then f is a constant.*

Proof: Let $t \in \mathbb{R}$ and consider $h(t) = f(x + t(y-x)) - f(x)$. Then by the chain rule,

$$h'(t) = f'(x + t(y-x))(y-x) = 0$$

and so by Corollary 7.8.4 h is a constant. In particular,

$$h(1) = f(y) - f(x) = h(0) = 0$$

which shows f is constant since x, y are arbitrary. This proves the corollary.

Corollary 7.8.6 *Suppose $f'(x) > 0$ for all $x \in (a,b)$ where $a \geq -\infty$ and $b \leq \infty$. Then f is strictly increasing on (a,b). That is, if $x < y$, then $f(x) < f(y)$. If $f'(x) \geq 0$, then f is increasing in the sense that whenever $x < y$ it follows that $f(x) \leq f(y)$.*

Proof: Let $x < y$. Then by the mean value theorem, there exists $z \in (x,y)$ such that

$$0 < f'(z) = \frac{f(y) - f(x)}{y - x}.$$

Since $y > x$, it follows $f(y) > f(x)$ as claimed. Replacing $<$ by $\leq$ in the above equation and repeating the argument gives the second claim.

Corollary 7.8.7 *Suppose $f'(x) < 0$ for all $x \in (a,b)$ where $a \geq -\infty$ and $b \leq \infty$. Then f is strictly decreasing on (a,b). That is, if $x < y$, then $f(x) > f(y)$. If $f'(x) \leq 0$, then f is decreasing in the sense that for $x < y$, it follows that $f(x) \geq f(y)$*

Proof: Let $x < y$. Then by the mean value theorem, there exists $z \in (x,y)$ such that

$$0 > f'(z) = \frac{f(y) - f(x)}{y - x}.$$

Since $y > x$, it follows $f(y) < f(x)$ as claimed. The second claim is similar except instead of a strict inequality in the above formula, you put $\geq$.

7.9 Exercises

1. Sally drives her Saturn over the 110 mile toll road in exactly 1.3 hours. The speed limit on this toll road is 70 miles per hour and the fine for speeding is 10 dollars per mile per hour over the speed limit. How much should Sally pay?

2. Two cars are careening down a freeway weaving in and out of traffic. Car A passes car B and then car B passes car A as the driver makes obscene gestures. This infuriates the driver of car A who passes car B while firing his handgun at the driver of car B. Show there are at least two times when both cars have the same speed. Then show there exists at least one time when they have the same acceleration. The acceleration is the derivative of the velocity.

3. Show the cubic function, $f(x) = 5x^3 + 7x - 18$ has only one real zero.

4. Suppose $f(x) = x^7 + |x| + x - 12$. How many solutions are there to the equation, $f(x) = 0$?

5. Let $f(x) = |x - 7| + (x - 7)^2 - 2$ on the interval $[6, 8]$. Then $f(6) = 0 = f(8)$. Does it follow from Rolle's theorem that there exists $c \in (6, 8)$ such that $f'(c) = 0$? Explain your answer.

6. Suppose f and g are differentiable functions defined on $\mathbb{R}$. Suppose also that it is known that $|f'(x)| > |g'(x)|$ for all x and that $|f'(t)| > 0$ for all t. Show that whenever $x \neq y$, it follows $|f(x) - f(y)| > |g(x) - g(y)|$. **Hint:** Use the Cauchy mean value theorem, Theorem 7.8.2.

7. Show that, like continuous functions, functions which are derivatives have the intermediate value property. This means that if $f'(a) < 0 < f'(b)$ then there exists $x \in (a, b)$ such that $f'(x) = 0$. **Hint:** Argue the minimum value of f occurs at an interior point of $[a, b]$.

8. Find an example of a function which has a derivative at every point but such that the derivative is not everywhere continuous.

9. Consider the function
$$f(x) \equiv \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases}.$$
Is it possible that this function could be the derivative of some function? Why?

10. Suppose $a \in I$, an open interval and that a function f, defined on I has $n+1$ derivatives. Then for each $m \leq n$ the following formula holds for $x \in I$.
$$f(x) = \sum_{k=0}^{m} f^{(k)}(a) \frac{(x-a)^k}{k!} + f^{(m+1)}(y) \frac{(x-a)^{m+1}}{(m+1)!} \tag{7.19}$$
where y is some point between x and a. Note that if $n = 0$, this reduces to the Lagrange form of the mean value theorem so the formula holds if $m = 0$. Suppose it holds for some $0 \leq m - 1 < n$. The task is to show then that it also holds for m. It will then follow that the above formula will hold for all $m \leq n$ as claimed. This formula is very important. The last term is called the Lagrange form of the remainder in Taylor series. Try to prove the formula using the following steps. If the formula holds for $m - 1$, then you can apply it to f'.
$$f'(x) - \sum_{k=0}^{m-1} \frac{f^{(k+1)}(a)}{k!} (x-a)^k = \frac{f^{(m+1)}(y)}{m!} (x-a)^m. \tag{7.20}$$

Now let $g(x) = f(x) - \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x-a)^k$ and let $h(x) = (x-a)^{m+1}$. Now from the Cauchy mean value theorem there exists z between x and a such that

$$\frac{g(x)-g(a)}{h(x)-h(a)} = \frac{g'(z)}{h'(z)} = \frac{f'(z) - \sum_{k=1}^{m} \frac{f^{(k)}(a)}{(k-1)!}(z-a)^{k-1}}{(m+1)(z-a)^m}.$$

Now explain why $\sum_{k=1}^{m} \frac{f^{(k)}(a)}{(k-1)!}(z-a)^{k-1} = \sum_{k=0}^{m-1} \frac{f^{(k+1)}(a)}{k!}(z-a)^k$ and then use 7.20. Explain why this implies

$$\frac{f(x) - \sum_{k=0}^{m} \frac{f^{(k)}(a)}{k!}(x-a)^k}{(x-a)^{m+1}} = \frac{\frac{f^{(m+1)}(y)}{m!}(z-a)^m}{(m+1)(z-a)^m}$$

which yields the desired formula for m. A way to think of 7.19 is as a generalized mean value theorem. Note that the key result which made it work was the Cauchy mean value theorem.

11. Now here is another way to obtain the above approximation. Fix c, x in (a,b) an interval on which f has $n+1$ derivatives. Let K be a number, depending on c, x such that

$$f(x) - \left(f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + K(x-c)^{n+1} \right) = 0$$

Now the idea is to find K. To do this, let

$$F(t) = f(x) - \left(f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + K(x-t)^{n+1} \right)$$

Then $F(x) = F(c) = 0$. Therefore, by Rolle's theorem there exists z between a and x such that $F'(z) = 0$. Do the differentiation and solve for K.

12. Let f be a real continuous function defined on the interval $[0,1]$. Also suppose $f(0) = 0$ and $f(1) = 1$ and $f'(t)$ exists for all $t \in (0,1)$. Show there exists n distinct points $\{s_i\}_{i=1}^n$ of the interval such that

$$\sum_{i=1}^{n} f'(s_i) = n.$$

Hint: Consider the mean value theorem applied to successive pairs in the following sum.

$$f\left(\frac{1}{3}\right) - f(0) + f\left(\frac{2}{3}\right) - f\left(\frac{1}{3}\right) + f(1) - f\left(\frac{2}{3}\right)$$

13. Now suppose $f : [0,1] \to \mathbb{R}$ and $f(0) = 0$ while $f(1) = 1$. Show there are distinct points $\{s_i\}_{i=1}^n \subseteq (0,1)$ such that

$$\sum_{i=1}^{n} (f'(s_i))^{-1} = n.$$

Hint: Let $0 = t_0 < t_1 < \cdots < t_n = 1$ and pick $x_i \in f^{-1}(t_i)$ such that these x_i are increasing and $x_n = 1, x_0 = 0$. Explain why you can do this. Then argue

$$t_{i+1} - t_i = f(x_{i+1}) - f(x_i) = f'(s_i)(x_{i+1} - x_i)$$

and so

$$\frac{x_{i+1} - x_i}{t_{i+1} - t_i} = \frac{1}{f'(s_i)}$$

Now choose the t_i to be equally spaced.

7.10 Derivatives Of Inverse Functions

It happens that if f is a differentiable one to one function defined on an interval, $[a,b]$, and $f'(x)$ exists and is non zero then the inverse function, f^{-1} has a derivative at the point $f(x)$. Recall that f^{-1} is defined according to the formula

$$f^{-1}(f(x)) = x.$$

Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Recall from Theorem 7.5.1

$$f'(a) \equiv \lim_{x\to a+}\frac{f(x)-f(a)}{x-a},\ f'(b) \equiv \lim_{x\to b-}\frac{f(x)-f(b)}{x-b}.$$

Recall the notation $x\to a+$ means that only $x>a$ are considered in the definition of limit, the notation $x\to b-$ defined similarly. Thus, this definition includes the derivative of f at the endpoints of the interval and to save notation,

$$f'(x_1) \equiv \lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1}$$

where it is understood that x is always in $[a,b]$.

Theorem 7.10.1 *Let $f:[a,b]\to\mathbb{R}$ be continuous and one to one. Suppose $f'(x_1)$ exists for some $x_1\in[a,b]$ and $f'(x_1)\neq 0$. Then $\left(f^{-1}\right)'(f(x_1))$ exists and is given by the formula, $\left(f^{-1}\right)'(f(x_1)) = \frac{1}{f'(x_1)}$.*

Proof: By Lemma 6.4.3, and Corollary 6.4.4 on Page 100 f is either strictly increasing or strictly decreasing and f^{-1} is continuous. Therefore there exists $\eta>0$ such that if $0<|f(x_1)-f(x)|<\eta$, then

$$0<|x_1-x| = \left|f^{-1}(f(x_1))-f^{-1}(f(x))\right| < \delta$$

where δ is small enough that for $0<|x_1-x|<\delta$,

$$\left|\frac{x-x_1}{f(x)-f(x_1)}-\frac{1}{f'(x_1)}\right|<\varepsilon.$$

It follows that if $0<|f(x_1)-f(x)|<\eta$,

$$\left|\frac{f^{-1}(f(x))-f^{-1}(f(x_1))}{f(x)-f(x_1)}-\frac{1}{f'(x_1)}\right| = \left|\frac{x-x_1}{f(x)-f(x_1)}-\frac{1}{f'(x_1)}\right|<\varepsilon$$

Therefore, since $\varepsilon>0$ is arbitrary,

$$\lim_{y\to f(x_1)}\frac{f^{-1}(y)-f^{-1}(f(x_1))}{y-f(x_1)} = \frac{1}{f'(x_1)}$$

and this proves the theorem.

The following obvious corollary comes from the above by not bothering with end points.

Corollary 7.10.2 *Let $f:(a,b)\to\mathbb{R}$, where $-\infty\le a<b\le\infty$ be continuous and one to one. Suppose $f'(x_1)$ exists for some $x_1\in(a,b)$ and $f'(x_1)\neq 0$. Then $\left(f^{-1}\right)'(f(x_1))$ exists and is given by the formula, $\left(f^{-1}\right)'(f(x_1)) = \frac{1}{f'(x_1)}$.*

This is one of those theorems which is very easy to remember if you neglect the difficult questions and simply focus on formal manipulations. Consider the following.

$$f^{-1}(f(x)) = x.$$

Now use the chain rule on both sides to write

$$\left(f^{-1}\right)'(f(x))\, f'(x) = 1,$$

and then divide both sides by $f'(x)$ to obtain

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}.$$

Of course this gives the conclusion of the above theorem rather effortlessly and it is formal manipulations like this which aid in remembering formulas such as the one given in the theorem.

Example 7.10.3 *Let $f(x) = 1+x^2+x^3+7$. Show that f has an inverse and find $\left(f^{-1}\right)'(8)$.*

I am not able to find a formula for the inverse function. This is typical in useful applications so you need to get used to this idea. The methods of algebra are insufficient to solve hard problems in analysis. You need something more. The question is to determine whether f has an inverse. To do this,

$$f'(x) = 2x + 3x^2 + 7 > 0$$

By Corollary 7.8.6 on Page 133, this function is strictly increasing on $\mathbb{R}$ and so it has an inverse function although I have no idea how to find an explicit formula for this inverse function. However, I can see that $f(0) = 8$ and so by the formula for the derivative of an inverse function,

$$\left(f^{-1}\right)'(8) = \left(f^{-1}\right)'(f(0)) = \frac{1}{f'(0)} = \frac{1}{7}.$$

7.11 Derivatives And Limits Of Sequences

When you have a function which is a limit of a sequence of functions, when can you say the derivative of the limit function is the limit of the derivatives of the functions in the sequence? The following theorem seems to be one of the best results available. It is based on the mean value theorem. First of all, recall Definition 6.8.5 on Page 105 listed here for convenience.

Definition 7.11.1 *Let $\{f_n\}$ be a sequence of functions defined on D. Then $\{f_n\}$ is said to converge uniformly to f if it converges pointwise to f and for every $\varepsilon > 0$ there exists N such that for all $n \geq N$*

$$|f(x) - f_n(x)| < \varepsilon$$

for all $x \in D$.

To save on notation, denote by

$$||k|| \equiv \sup\{|k(\xi)| : \xi \in D\}.$$

Then

$$||k + l|| \leq ||k|| + ||l|| \tag{7.21}$$

because for each $\xi \in D$,

$$|k(\xi) + l(\xi)| \leq ||k|| + ||l||$$

and taking sup yields 7.21. From the definition of uniform convergence, you see that f_n converges uniformly to f is the same as saying

$$\lim_{n\to\infty} ||f_n - f|| = 0.$$

Now here is the theorem.

Theorem 7.11.2 *Let (a,b) be a finite open interval and let $f_k : (a,b) \to \mathbb{R}$ be differentiable and suppose there exists $x_0 \in (a,b)$ such that*

$$\{f_k(x_0)\} \text{ converges,}$$

$$\{f_k'\} \text{ converges uniformly to a function } g \text{ on } (a,b).$$

Then there exists a function, f defined on (a,b) such that

$$f_k \to f \text{ uniformly,}$$

and

$$f' = g.$$

Proof: Let $c \in (a,b)$ and define

$$g_n(x) \equiv \begin{cases} \frac{f_n(x) - f_n(c)}{x - c} & \text{if } x \neq c \\ f_n'(c) & \text{if } x = c \end{cases}.$$

Note that g_n really depends on c but this is suppressed in the interest of simpler notation.

Claim 1: For each c, g_n converges uniformly to a continuous function, h, on (a,b).

Proof: First note that each g_n is continuous. Let $x \neq c$. Then by the mean value theorem,

$$\begin{aligned} & |g_n(x) - g_m(x)| \\ = & \left|\frac{f_n(x) - f_m(x) - (f_n(c) - f_m(c))}{x - c}\right| \\ = & |f_n'(\xi) - f_m'(\xi)| \leq |f_n'(\xi) - g(\xi)| + |g(\xi) - f_m'(\xi)| \\ \leq & ||f_n' - g|| + ||f_m' - g|| \end{aligned}$$

By the assumption that $\{f_n'\}$ converges uniformly to g, it follows each of the last two terms converges to 0 as $n, m \to \infty$. Let $h(x)$ be the name of $\lim_{m\to\infty} g_m(x)$ which exists because of completeness. Letting $m \to \infty$ in the above yields for all $x \in (a,b)$,

$$|g_n(x) - h(x)| \leq ||f_n' - g||$$

Thus the convergence of g_n to h is uniform as claimed and consequently h is continuous by Theorem 6.8.6. This proves the first claim.

Claim 2: f_n converges uniformly to a function, f.

Proof: From the definition of g_n in the case where $c = x_0$,

$$f_n(x) - f_n(x_0) = g_n(x)(x - x_0). \tag{7.22}$$

It follows from this and the assumption that $f_n(x_0)$ converges, that $f_n(x)$ converges for each x. Let $f(x)$ denote the thing to which it converges. Thus

$$f(x) - f(x_0) = h(x)(x - x_0) \tag{7.23}$$

and so, subtracting 7.22 and 7.23 yields

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x_0) - f_n(x_0)| + |(h(x) - g_n(x))(x - x_0)| \\ &\leq |f(x_0) - f_n(x_0)| + ||h - g_n||(b - a). \end{aligned}$$

and so

$$||f_n - f|| \leq ||g_n - h|| \, |b - a| + |f_n(x_0) - f(x_0)|,$$

showing that $f_n \to f$ uniformly as claimed.

Now to complete the proof of the theorem,

$$\frac{f(x) - f(c)}{x - c} = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x - c} = \lim_{n\to\infty} g_n(x) = h(x).$$

Since h is continuous,

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} h(x) = h(c).$$

It remains to verify $h(c) = g(c)$. This is easy because $g_n(c) \equiv f_n'(c) \to g(c)$ and $g_n(c) \to h(c)$. This proves the theorem.

7.12 Exercises

1. It was shown earlier that the n^{th} root of a positive number exists whenever n is a positive integer. Let $y = x^{1/n}$. Prove $y'(x) = \frac{1}{n}x^{(1/n)-1}$.

2. Now for positive x and p, q positive integers, $y = x^{p/q}$ is defined by $y = \sqrt[q]{x^p}$. Find and prove a formula for dy/dx.

3. For $1 \geq x \geq 0$, and $p \geq 1$, show that $(1 - x)^p \geq 1 - px$. **Hint:** This can be done using the mean value theorem. Define $f(x) \equiv (1 - x)^p - 1 + px$ and show that $f(0) = 0$ while $f'(x) \geq 0$ for all $x \in (0, 1)$.

4. Using the result of Problem 3 establish Raabe's Test, an interesting variation on the ratio test. This test says the following. Suppose there exists a constant, C and a number p such that
$$\left|\frac{a_{k+1}}{a_k}\right| \leq 1 - \frac{p}{k + C}$$
for all k large enough. Then if $p > 1$, it follows that $\sum_{k=1}^{\infty} a_k$ converges absolutely. **Hint:** Let $b_k \equiv k - 1 + C$ and note that for all k large enough, $b_k > 1$. Now conclude that there exists an integer, k_0 such that $b_{k_0} > 1$ and for all $k \geq k_0$ the given inequality above holds. Use Problem 3 to conclude that
$$\left|\frac{a_{k+1}}{a_k}\right| \leq 1 - \frac{p}{k + C} \leq \left(1 - \frac{1}{k + C}\right)^p = \left(\frac{b_k}{b_{k+1}}\right)^p$$
showing that $|a_k| b_k^p$ is decreasing for $k \geq k_0$. Thus $|a_k| \leq C/b_k^p = C/(k - 1 + C)^p$. Now use comparison theorems and the p series to obtain the conclusion of the theorem.

5. The graph of a function, $y = f(x)$ is said to be concave up or more simply "convex" if whenever (x_1, y_1) and (x_2, y_2) are two points such that $y_i \geq f(x_i)$, it follows that for each point, (x, y) on the straight line segment joining (x_1, y_1) and (x_2, y_2), $y \geq f(x)$. Show that if f is twice differentiable on an open interval, (a, b) and $f''(x) > 0$, then the graph of f is convex.

6. Show that if the graph of a function, f defined on an interval (a, b) is convex, then if f' exists on (a, b), it must be the case that f' is an increasing function. Note you do not know the second derivative exists.

7. Convex functions defined in Problem 5 have a very interesting property. Suppose $\{a_i\}_{i=1}^n$ are all nonnegative and suppose ϕ is a convex function defined on $\mathbb{R}$. Then

$$\phi\left(\sum_{k=1}^n a_k x_k\right) \leq \sum_{k=1}^n a_k \phi(x_k).$$

Verify this interesting inequality.

8. If ϕ is a convex function defined on $\mathbb{R}$, show that ϕ must be continuous at every point.

9. Prove the second derivative test. If $f'(x) = 0$ at $x \in (a, b)$, an interval on which f is defined and both f', f'' exist on this interval, then if $f''(x) > 0$, it follows f has a local minimum at x and if $f''(x) < 0$, then f has a local maximum at x. Show that if $f''(x) = 0$ no conclusion about the nature of the critical point can be drawn. It might be a local minimum, local maximum or neither.

10. Recall the Bernstein polynomials which were used to prove the Weierstrass approximation theorem. For f a continuous function on $[0, 1]$,

$$p_n(x) = \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

It was shown these converge uniformly to f on $[0, 1]$. Now suppose f' exists and is continuous on $[0, 1]$. Show p_n' converges uniformly to f' on $[0, 1]$. **Hint:** Differentiate the above formula and massage to finally get

$$p_n'(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{1/n}\right) x^k (1-x)^{n-1-k}.$$

Then form the $(n-1)$ Bernstein polynomial for f' and show the two are uniformly close. You will need to estimate an expression of the form

$$f'\left(\frac{k}{n-1}\right) - \frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{1/n}$$

which will be easy to do because of the mean value theorem and uniform continuity of f'.

11. In contrast to Problem 10, consider the sequence of functions

$$\{f_n(x)\}_{n=1}^\infty = \left\{\frac{x}{1+nx^2}\right\}_{n=1}^\infty.$$

Show it converges uniformly to $f(x) \equiv 0$. However, $f_n'(0)$ converges to 1, not $f'(0)$. **Hint:** To show the first part, find the value of x which maximizes the function $\left|\frac{x}{1+nx^2}\right|$. You know how to do this. Then plug it in and you will have an estimate sufficient to verify uniform convergence.

12. Let f be a real invertible continuous function defined on the interval $[0,1]$. Also suppose $f(0)=0$ and $f(1)=1$ and $f'(t)$ exists for all $t\in(0,1)$. Show there exists n distinct points $\{s_i\}_{i=1}^n$ of the interval such that

$$\sum_{i=1}^{n}\frac{1}{f'(s_i)}=n.$$

Hint: Consider using Problem 12 on Page 135 applied to the inverse function.

Power Series

8.1 Functions Defined In Terms Of Series

It is time to consider functions other than polynomials. In particular it is time to give a mathematically acceptable definition of functions like $e^x, \sin(x)$ and $\cos(x)$. It has been assumed these functions are known from beginning calculus but this is a pretence. Most students who take calculus come through it without a complete understanding of the circular functions. This is because of the reliance on plane geometry in defining them. Fortunately, these functions can be completely understood in terms of power series rather than wretched plane geometry. The exponential function can also be defined in a simple manner using power series.

Definition 8.1.1 *Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of numbers. The expression,*

$$\sum_{k=0}^{\infty} a_k (x-a)^k \tag{8.1}$$

is called a Taylor series centered at a. This is also called a power series centered at a. It is understood that x and $a \in \mathbb{F}$, that is, either $\mathbb{C}$ or $\mathbb{R}$.

In the above definition, x is a variable. Thus you can put in various values of x and ask whether the resulting series of numbers converges. Defining D to be the set of all values of x such that the resulting series does converge, define a new function, f defined on D having values in $\mathbb{F}$ as

$$f(x) \equiv \sum_{k=0}^{\infty} a_k (x-a)^k .$$

This might be a totally new function, one which has no name. Nevertheless, much can be said about such functions. The following lemma is fundamental in considering the form of D which always turns out to be of the form $B(a,r)$ along with possibly some points, z such that $|z-a| = r$. First here is a simple lemma which will be useful.

Lemma 8.1.2 $\lim_{n\to\infty} n^{1/n} = 1$.

Proof: It is clear $n^{1/n} \geq 1$. Let $n^{1/n} = 1 + e_n$ where $0 \leq e_n$. Then raising both sides to the n^{th} power for $n > 1$ and using the binomial theorem,

$$\begin{aligned} n &= (1+e_n)^n = \sum_{k=0}^{n} \binom{n}{k} e_n^k \geq 1 + ne_n + (n(n-1)/2) e_n^2 \\ &\geq (n(n-1)/2) e_n^2 \end{aligned}$$

Thus

$$0 \leq e_n^2 \leq \frac{n}{n(n-1)} = \frac{1}{n-1}$$

From this the desired result follows because

$$\left|n^{1/n} - 1\right| = e_n \leq \frac{1}{\sqrt{n-1}}.$$

Theorem 8.1.3 *Let $\sum_{k=0}^{\infty} a_k (x-a)^k$ be a Taylor series. Then there exists $r \leq \infty$ such that the Taylor series converges absolutely if $|x-a| < r$. Furthermore, if $|x-a| > r$, the Taylor series diverges.*

Proof: Note

$$\limsup_{k\to\infty} \left|a_k (x-a)^k\right|^{1/k} = \limsup_{k\to\infty} |a_k|^{1/k} |x-a|.$$

Then by the root test, the series converges absolutely if

$$|x-a| \limsup_{k\to\infty} |a_k|^{1/k} < 1$$

and diverges if

$$|x-a| \limsup_{k\to\infty} |a_k|^{1/k} > 1.$$

Thus define

$$r \equiv \begin{cases} 1/\limsup_{k\to\infty} |a_k|^{1/k} \text{ if } \infty > \limsup_{k\to\infty} |a_k|^{1/k} > 0 \\ \infty \text{ if } \limsup_{k\to\infty} |a_k|^{1/k} = 0 \\ 0 \text{ if } \limsup_{k\to\infty} |a_k|^{1/k} = \infty \end{cases}$$

This proves the theorem.

Note that the radius of convergence, r is given by

$$\limsup_{k\to\infty} |a_k|^{1/k} r = 1$$

Definition 8.1.4 *The number in the above theorem is called the radius of convergence and the set on which convergence takes place is called the disc of convergence.*

Now the theorem was proved using the root test but often you use the ratio test to actually find the interval of convergence. This kind of thing is typical in math so get used to it. The proof of a theorem does not always yield a way to find the thing the theorem speaks about. The above is an existence theorem. There exists an interval of convergence from the above theorem. You find it in specific cases any way that is most convenient.

Example 8.1.5 *Find the disc of convergence of the Taylor series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.*

Use Corollary 5.3.10.

$$\lim_{n\to\infty} \left(\frac{|x|^n}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}} = |x|$$

because $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and so if $|x| < 1$ the series converges. The points satisfying $|z| = 1$ require special attention. When $x = 1$ the series diverges because it reduces to $\sum_{n=1}^{\infty} \frac{1}{n}$. At $x = -1$ the series converges because it reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and the alternating series test applies and gives convergence. What of the other numbers z satisfying $|z| = 1$? It turns out this series will converge at all these numbers by the Dirichlet test. However, this will require something like De Moivre's theorem and this has not yet been presented carefully.

Example 8.1.6 *Find the radius of convergence of* $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$.

Apply the ratio test. Taking the ratio of the absolute values of the $(n+1)^{th}$ and the n^{th} terms

$$\frac{\frac{(n+1)^{(n+1)}}{(n+1)n!} |x|^{n+1}}{\frac{n^n}{n!} |x|^n} = (n+1)^n |x| n^{-n} = |x| \left(1 + \frac{1}{n}\right)^n \to |x| e$$

Therefore the series converges absolutely if $|x| e < 1$ and diverges if $|x| e > 1$. Consequently, $r = 1/e$.

8.2 Operations On Power Series

It is desirable to be able to differentiate and multiply power series. The following theorem says you can differentiate power series in the most natural way on the interval of convergence, just as you would differentiate a polynomial. This theorem may seem obvious, but it is a serious mistake to think this. You usually cannot differentiate an infinite series whose terms are functions even if the functions are themselves polynomials. The following is special and pertains to power series. It is another example of the interchange of two limits, in this case, the limit involved in taking the derivative and the limit of the sequence of finite sums.

When you formally differentiate a series term by term, the result is called the derived series.

Theorem 8.2.1 *Let* $\sum_{n=0}^{\infty} a_n (x-a)^n$ *be a Taylor series having radius of convergence* $R > 0$ *and let*

$$f(x) \equiv \sum_{n=0}^{\infty} a_n (x-a)^n \tag{8.2}$$

for $|x-a| < R$. *Then*

$$f'(x) = \sum_{n=0}^{\infty} a_n n (x-a)^{n-1} = \sum_{n=1}^{\infty} a_n n (x-a)^{n-1} \tag{8.3}$$

and this new differentiated power series, the derived series, has radius of convergence equal to R.

Proof: Consider $g(z) = \sum_{k=2}^{\infty} a_k k (z-a)^{k-1}$ on $B(a, R)$ where R is the radius of convergence defined above. Let $r_1 < r < R$. Then letting $|z-a| < r_1$ and $h < r - r_1$,

$$\begin{aligned}
&\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \\
\leq & \sum_{k=2}^{\infty} |a_k| \left| \frac{(z+h-a)^k - (z-a)^k}{h} - k(z-a)^{k-1} \right| \\
\leq & \sum_{k=2}^{\infty} |a_k| \left| \frac{1}{h} \left(\sum_{i=0}^{k} \binom{k}{i} (z-a)^{k-i} h^i - (z-a)^k \right) - k(z-a)^{k-1} \right| \\
= & \sum_{k=2}^{\infty} |a_k| \left| \frac{1}{h} \left(\sum_{i=1}^{k} \binom{k}{i} (z-a)^{k-i} h^i \right) - k(z-a)^{k-1} \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=2}^{\infty} |a_k| \left| \left(\sum_{i=2}^{k} \binom{k}{i} (z-a)^{k-i} h^{i-1} \right) \right| \\
&\leq |h| \sum_{k=2}^{\infty} |a_k| \left(\sum_{i=0}^{k-2} \binom{k}{i+2} |z-a|^{k-2-i} |h|^i \right) \\
&= |h| \sum_{k=2}^{\infty} |a_k| \left(\sum_{i=0}^{k-2} \binom{k-2}{i} \frac{k(k-1)}{(i+2)(i+1)} |z-a|^{k-2-i} |h|^i \right) \\
&\leq |h| \sum_{k=2}^{\infty} |a_k| \frac{k(k-1)}{2} \left(\sum_{i=0}^{k-2} \binom{k-2}{i} |z-a|^{k-2-i} |h|^i \right) \\
&= |h| \sum_{k=2}^{\infty} |a_k| \frac{k(k-1)}{2} (|z-a| + |h|)^{k-2} < |h| \sum_{k=2}^{\infty} |a_k| \frac{k(k-1)}{2} r^{k-2}.
\end{aligned}
$$

Then

$$\limsup_{k\to\infty} \left(|a_k| \frac{k(k-1)}{2} r^{k-2} \right)^{1/k} = \limsup_{k\to\infty} |a_k|^{1/k} r < 1$$

because $r < R$ and R is defined by

$$R = 1/ \limsup_{k\to\infty} |a_k|^{1/k}.$$

Thus, by the root test,

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \leq C |h| \tag{8.4}$$

where

$$C = \sum_{k=2}^{\infty} |a_k| \frac{k(k-1)}{2} r^{k-2}.$$

therefore, $g(z) = f'(z)$ because by 8.4,

$$f(z+h) - f(z) - g(z) h = o(h).$$

This proves $g(z) = f'(z)$ for any $|z| < R$. What about the radius of convergence of g? It is the same as the radius of convergence of f because

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} k^{1/k} |a_k|^{1/k}$$

and

$$\limsup_{k\to\infty} \left(|z-a|^{k-1} \right)^{1/k} = \limsup_{k\to\infty} \left(|z-a|^{k} \right)^{1/k} = |z-a|.$$

This proves the theorem.

As an immediate corollary, it is possible to characterize the coefficients of a Taylor series.

Corollary 8.2.2 *Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be a Taylor series with radius of convergence $r > 0$ and let*

$$f(x) \equiv \sum_{n=0}^{\infty} a_n (x-a)^n. \tag{8.5}$$

Then

$$a_n = \frac{f^{(n)}(a)}{n!}. \tag{8.6}$$

Proof: From 8.5, $f(a) = a_0 \equiv f^{(0)}(a)/0!$. From Theorem 8.2.1,

$$f'(x) = \sum_{n=1}^{\infty} a_n n (x-a)^{n-1} = a_1 + \sum_{n=2}^{\infty} a_n n (x-a)^{n-1}.$$

Now let $x = a$ and obtain that $f'(a) = a_1 = f'(a)/1!$. Next use Theorem 8.2.1 again to take the second derivative and obtain

$$f''(x) = 2a_2 + \sum_{n=3}^{\infty} a_n n (n-1)(x-a)^{n-2}$$

let $x = a$ in this equation and obtain $a_2 = f''(a)/2 = f''(a)/2!$. Continuing this way proves the corollary.

This also shows the coefficients of a Taylor series are unique. That is, if

$$\sum_{k=0}^{\infty} a_k (x-a)^k = \sum_{k=0}^{\infty} b_k (x-a)^k$$

for all x in some open set containing a, then $a_k = b_k$ for all k.

Example 8.2.3 *Find the sum* $\sum_{k=1}^{\infty} k2^{-k}$.

It may not be obvious what this sum equals but with the above theorem it is easy to find. From the formula for the sum of a geometric series, $\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$ if $|t| < 1$. Differentiate both sides to obtain

$$(1-t)^{-2} = \sum_{k=1}^{\infty} kt^{k-1}$$

whenever $|t| < 1$. Let $t = 1/2$. Then

$$4 = \frac{1}{(1-(1/2))^2} = \sum_{k=1}^{\infty} k2^{-(k-1)}$$

and so if you multiply both sides by 2^{-1},

$$2 = \sum_{k=1}^{\infty} k2^{-k}.$$

8.3 The Special Functions Of Elementary Calculus

8.3.1 The Functions, sin, cos, exp

With this material on power series, it becomes possible to give an understandable treatment of the exponential function, exp and the circular functions, sin and cos.

Definition 8.3.1 *Define for all* $x \in \mathbb{F}$

$$\sin(x) \equiv \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad \cos(x) \equiv \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\exp(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Observation 8.3.2 *The above series converge for all $x \in \mathbb{F}$. This is most easily seen using the ratio test. Consider the series for* $\sin(x)$ *first. By the ratio test the series converges whenever*

$$\lim_{k\to\infty} \frac{\frac{|x|^{2k+3}}{(2k+3)!}}{\frac{|x|^{2k+1}}{(2k+1)!}} = \lim_{k\to\infty} \frac{1}{(2k+3)(2k+1)} |x|^2$$

is less than 1. However, this limit equals 0 for any x and so the series converges for all x. The verification of convergence for the other two series is left for you to do and is no harder.

Now that $\sin(x)$ and $\cos(x)$ have been defined, the properties of these functions must be considered. First, here are some lemmas.

Lemma 8.3.3 *For fixed $y \in \mathbb{F}$, the function of x given by*

$$x \to \sin(x+y)$$

may be written in the form

$$\sum_{k=0}^{\infty} a_k(y) x^k$$

where the series converges on all of $\mathbb{F}$. The same is true of the function

$$x \to \cos(x+y).$$

Proof: Let

$$l(0) = 0, l(1) = 1, l(2) = 0, l(3) = -1,$$

etc. Thus $l(\text{even}) = 0$ and $l(2n+1) = (-1)^n$. Then from the definition of sin,

$$\begin{aligned} \sin(x+y) &= \sum_{n=0}^{\infty} l(n) \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{l(n)}{n!}\binom{n}{k} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{l(n)}{(n-k)!k!} x^k y^{n-k}. \end{aligned} \tag{8.7}$$

Now consider the absolute convergence of this double sum.

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \left| \frac{l(n)}{(n-k)!k!} x^k y^{n-k} \right| &\leq \sum_{n=0}^{\infty} \frac{|x|^k}{k!} \sum_{k=0}^{n} \frac{1}{(n-k)!} |y|^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{|x|^k}{k!} \sum_{l=0}^{n} \frac{|y|^l}{l!} \\ &\leq \sum_{n=0}^{\infty} \frac{|x|^k}{k!} \sum_{l=0}^{\infty} \frac{|y|^l}{l!} < \infty \end{aligned}$$

by an application of the ratio test to the two series in the product. Therefore, by Theorem 5.4.6, the order of summation in 8.7 may be reversed and this yields

$$\begin{aligned} \sin(x+y) &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{l(n)}{(n-k)!k!} x^k y^{n-k} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=k}^{\infty} \frac{l(n)}{(n-k)!} y^{n-k} \\ &= \sum_{k=0}^{\infty} a_k(y) \frac{x^k}{k!} \end{aligned}$$

where by Theorem 5.4.6, all infinite series in the above converge. The claim about $x \to \cos(x+y)$ follows from similar reasoning. This proves the lemma.

Lemma 8.3.4 *Suppose y is an $\mathbb{F}$ valued convergent power series which converges on some open ball containing 0 and it solves the initial value problem,*

$$y'' + y = 0,\ y(0) = 0, y'(0) = 0$$

Then $y(x) = 0$.

Proof: By assumption,

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{8.8}$$

Then by Theorem 8.2.1

$$y'(x) = \sum_{k=1}^{\infty} a_k k x^{k-1},\ y''(x) = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

Thus,

$$y''(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k \tag{8.9}$$

The conditions $y(0) = 0$ and $y'(0) = 0$ imply $a_0 = a_1 = 0$ because by Corollary 8.2.2 $a_0 = y(0), a_1 = y'(0)$. Now using the equation and 8.8 and 8.9, it follows

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k\right) x^k = 0$$

which by Corollary 8.2.2 requires that for all k,

$$a_{k+2} = \frac{-a_k}{(k+2)(k+1)}.$$

Since $a_0 = a_1 = 0$, this implies $a_k = 0$ for all k. Thus $y = 0$ and this proves the lemma.

Theorem 8.3.5 $\sin'(x) = \cos(x)$ *and* $\cos'(x) = -\sin(x)$. *Also* $\cos(0) = 1, \sin(0) = 0$ *and*

$$\cos^2(x) + \sin^2(x) = 1 \tag{8.10}$$

for all x. Also $\sin(-x) = -\sin(x)$ *while* $\cos(-x) = \cos(x)$ *and the usual trig. identities hold,*

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x) \tag{8.11}$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{8.12}$$

Proof: That $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$ follows right away from differentiating the power series term by term using Theorem 8.2.1. It follows from the series that $\cos(0) = 1$ and $\sin(0) = 0$ and $\sin(-x) = -\sin(x)$ while $\cos(-x) = \cos(x)$ because the series for $\sin(x)$ only involves odd powers of x while the series for $\cos(x)$ only involves even powers.

For $x \in \mathbb{C}$, let $f(x) = \cos^2(x) + \sin^2(x)$, it follows from what was just discussed that $f(0) = 1$. Also from the chain rule,

$$f'(x) = 2\cos(x)(-\sin(x)) + 2\sin(x)\cos(x) = 0$$

and so by Corollary 7.8.5, $f(x)$ is constant for all $x \in \mathbb{C}$. But $f(0) = 1$ so the constant can only be 1. Thus

$$\cos^2(x) + \sin^2(x) = 1$$

as claimed.

It only remains to verify the identities. By Lemma 8.3.3, $x \to \sin(x+y)$ is a power series converging on all of $\mathbb{F}$. Therefore, for fixed y,

$$g(x) \equiv \sin(x+y) - (\sin(x)\cos(y) + \sin(y)\cos(x))$$

is also a power series. Then $g(0) = \sin(y) - \sin(y) = 0$ and by the product rule and chain rule,

$$g'(x) = \cos(x+y) - [\cos(x)\cos(y) - \sin(y)\sin(x)]$$

Thus $g'(0) = 0$ also. Furthermore, it is routine to verify that $g'' + g = 0$. By Lemma 8.3.4, $g(x) = 0$ since both g and the 0 function satisfy the initial value problem of that lemma. To verify the other identity, fix y in 8.11 and differentiate with respect to x using the chain rule and the first part of the theorem. This proves the theorem.

Proposition 8.3.6 *The following important limits hold for $a, b \neq 0$.*

$$\lim_{x\to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}, \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

Proof: From the definition of $\sin(x)$ given above,

$$\begin{aligned} \frac{\sin(ax)}{bx} &= \frac{\sum_{k=0}^{\infty}(-1)^k \frac{(ax)^{2k+1}}{(2k+1)!}}{bx} = \frac{ax + \sum_{k=1}^{\infty}(-1)^k \frac{(ax)^{2k+1}}{(2k+1)!}}{bx} \\ &= \frac{a + \sum_{k=1}^{\infty}(-1)^k \frac{(ax)^{2k}}{(2k+1)!}}{b} \end{aligned}$$

Now

$$\begin{aligned} \left|\sum_{k=1}^{\infty}(-1)^k \frac{(ax)^{2k}}{(2k+1)!}\right| &\leq \sum_{k=1}^{\infty}|ax|^{2k} = \sum_{k=1}^{\infty}\left(|ax|^2\right)^k \\ &= \left(\frac{|ax|^2}{1-|ax|}\right) \end{aligned}$$

whenever $|ax| < 1$. Thus

$$\lim_{x\to 0}\sum_{k=1}^{\infty}(-1)^k \frac{(ax)^{2k}}{(2k+1)!} = 0$$

and so

$$\lim_{x\to 0} \frac{\sin(ax)}{bx} = \frac{a}{b}.$$

The other limit can be handled similarly.

It is possible to verify the functions are periodic. To do so, first consider the restrictions of these functions to $\mathbb{R}$ which yields a real valued function in each case. In this context, here is a simple lemma.

Lemma 8.3.7 *There exists a positive number, a, such that $\cos(a) = 0$.*

Proof: To prove this, note that $\cos(0) = 1$ and so if it is false, it must be the case that $\cos(x) > 0$ for all positive x since otherwise, it would follow from the intermediate value theorem there would exist a point, x where $\cos x = 0$. Assume this takes place. Then by Corollary 7.8.6 it would follow that $t \to \sin t$ is a strictly increasing function on $(0, \infty)$. Also note that $\sin(0) = 0$ and so $\sin(x) > 0$ for all $x > 0$. This is because, by the mean value theorem there exists $t \in (0, x)$ such that

$$\sin(x) = \sin(x) - \sin(0) = (\cos(t))(x - 0) > 0.$$

By 8.10, $|f(x)| \leq 1$ for $f = \cos$ and $\sin$. Let $0 < x < y$ where x. Then from the mean value theorem,

$$-\cos(y) - (-\cos(x)) = \sin(t)(y - x)$$

for some $t \in (x, y)$. Since $t \to \sin(t)$ is increasing, it follows

$$-\cos(y) - (-\cos(x)) = \sin(t)(y - x) \geq \sin(x)(y - x).$$

This contradicts the inequality $|\cos(y)| \leq 1$ for all y because the right side is unbounded as $y \to \infty$. This proves the lemma.

Theorem 8.3.8 *Both* cos *and* sin *are periodic.*

Proof: Define a number, π such that

$$\frac{\pi}{2} \equiv \inf\{x : x > 0 \text{ and } \cos(x) = 0\}$$

Then $\frac{\pi}{2} > 0$ because $\cos(0) = 1$ and cos is continuous. Also, $\cos\left(\frac{\pi}{2}\right) = 0$ because of continuity of cos. On $\left[0, \frac{\pi}{2}\right]$ cos is positive and so it follows sin is increasing on this interval. Therefore, from 8.10, $\sin\left(\frac{\pi}{2}\right) = 1$. Now from Theorem 8.3.5,

$$\cos(\pi) = \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\sin^2\left(\frac{\pi}{2}\right) = -1, \ \sin(\pi) = 0$$

Using Theorem 8.3.5 again,

$$\cos(2\pi) = \cos^2(\pi) = 1 = \cos(0),$$

and so $\sin(2\pi) = 0$. From Theorem 8.3.5,

$$\cos(x + 2\pi) = \cos(x)\cos(2\pi) - \sin(x)\sin(2\pi) = \cos(x)$$

Thus cos is periodic of period 2π. By Theorem 8.3.5,

$$\sin(x + 2\pi) = \sin(x)\cos(2\pi) + \cos(x)\sin(2\pi) = \sin(x)$$

Using 8.10, it follows sin is also periodic of period 2π. This proves the theorem.

Note that 2π is the smallest period for these functions. This can be seen by observing that the above theorem and proof imply that cos is positive on

$$\left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$$

and negative on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and that similar observations on sin are valid. Also, by considering where these functions are equal to 0, 1, and -1 along with where they are positive and negative, it follows that whenever $a^2 + b^2 = 1$, there exists a unique $t \in [0, 2\pi)$ such that

$\cos(t) = a$ and $\sin(t) = b$. For example, if a and b are both positive, then since cos is continuous and strictly decreases from 1 to 0 on $\left[0, \frac{\pi}{2}\right]$, it follows there exists a unique $t \in (0, \pi/2)$ such that $\cos(t) = a$. Since $b > 0$ and sin is positive on $(0, \pi/2)$, it follows $\sin(t) = b$. No other value of t in $[0,2\pi)$ will work since only on $(0, \pi/2)$ are both cos and sin positive. If $a > 0$ and $b < 0$ similar reasoning will show there exists a unique $t \in [0, 2\pi)$ with $\cos(t) = a$ and $\sin(t) = b$ and in this case, $t \in (3\pi/2, 2\pi)$. Other cases are similar and are left to the reader. Thus, every point on the unit circle is of the form $(\cos t, \sin t)$ for a unique $t \in [0, 2\pi)$.

This shows the unit circle is a smooth curve, however this notion will not be considered here.

Corollary 8.3.9 *For all $x \in \mathbb{F}$*

$$\sin(x + 2\pi) = \sin(x), \ \cos(x + 2\pi) = \cos(x)$$

Proof: Let $y(x) \equiv \sin(x + 2\pi) - \sin(x)$. Then from what has been shown above, $y'(0) = y(0) = 0$ and by Lemma 8.3.3 y is a power series. It is also clear from the above that $y'' + y = 0$. Therefore, from Lemma 8.3.4 $y = 0$. Differentiating the identity just obtained yields the second identity. This proves the corollary.

The functions, $x \to \sin(x)$ and $x \to \cos(x)$ have been defined on $\mathbb{F}$ which could be either $\mathbb{R}$ or $\mathbb{C}$. In the special case where $x \in \mathbb{R}$ are these the same as the circular functions you studied very sloppily in calculus and trigonometry? They are. Here is a simple lemma.

Lemma 8.3.10 *Suppose y is a real valued function defined on $\mathbb{R}$ which satisfies*

$$y'' + y = 0, \ y(0) = y'(0) = 0.$$

Then $y = 0$.

Proof: Multiply the equation by y' and use the chain rule to write

$$\frac{d}{dt}\left(\frac{1}{2}(y')^2 + \frac{1}{2}y^2\right) = 0.$$

Then by Corollary 7.8.5 $\frac{1}{2}(y')^2 + \frac{1}{2}y^2$ equals a constant. From the initial conditions, $y(0) = y'(0) = 0$, the constant can only be 0.

By Lemma 8.3.10 if it can be shown that $\sin(x)$ defined above and $\sin(x)$ studied in a beginning calculus class both satisfy the initial value problem

$$y'' + y = 0, \ y(0) = 0, y'(0) = 1$$

then they must be the same. However, if you remember anything from calculus you will realize $\sin(x)$ used there does satisfy the above initial value problem. If you don't remember anything from calculus, then it does not matter about harmonizing the functions. Just use the definition given above in terms of a power series. Similar considerations apply to cos.

Of course all the other trig. functions are defined as earlier. Thus

$$\tan x = \frac{\sin x}{\cos x}, \cot x \equiv \frac{\cos x}{\sin x}, \sec x \equiv \frac{1}{\cos x}, \csc x \equiv \frac{1}{\sin x}.$$

Using the techniques of differentiation, you can find the derivatives of all these.

Now it is time to consider the exponential function, $\exp(x)$ defined above.

Theorem 8.3.11 *The function,* exp *satisfies the following properties.*

1. $\exp(x) > 0$ *for all* $x \in \mathbb{R}$, $\lim_{x\to\infty} \exp(x) = \infty, \lim_{x\to-\infty} \exp(x) = 0$.

2. exp *is the unique solution to the initial value problem*

$$y' - y = 0, y(0) = 1 \tag{8.13}$$

3. *For all* $x, y \in \mathbb{F}$

$$\exp(x + y) = \exp(x)\exp(y) \tag{8.14}$$

4. exp *is one to one mapping* $\mathbb{R}$ *onto* $(0, \infty)$.

Proof: To begin with consider 8.14.

$$\exp(x)\exp(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

By an application of the ratio test, both series converge absolutely and so Theorem 5.4.8 and the binomial theorem can be applied to write the above product in the form

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} \frac{y^k}{k!} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (x + y)^n \equiv \exp(x + y). \end{aligned}$$

This verifies 8.14.

Next observe, it follows right away from the definition of $\exp(x)$ in terms of a power series that $\exp'(x) = \exp(x)$ so it satisfies the differential equation in the initial value problem of 8.13. It is obvious from the power series definition that it also satisfies the initial condition, $\exp(0) = 1$ which shows there exists a solution to 8.13. From 8.14, it follows $\exp(x)^{-1} = \exp(-x)$. Then letting y be a solution of 8.13, multiply the differential equation by $\exp(-x)$. By the chain rule this yields

$$\frac{d}{dx}(y(x)\exp(-x)) = 0.$$

Now by Corollary 7.8.5 $y(x)\exp(-x)$ is a constant. However, when $x = 0$ this expression is given to equal 1. Therefore, the constant is 1. Consequently, $y = \exp(x)$ and this proves uniqueness.

Now from the power series, it is obvious that $\exp(x) > 0$ if $x > 0$ and since $\exp(x)^{-1} = \exp(-x)$, it follows $\exp(-x)$ is also positive. Since $\exp(x) > \sum_{k=0}^{2} \frac{x^k}{k!}$, it is clear $\lim_{x\to\infty} \exp(x) = \infty$ and it follows from this that $\lim_{x\to-\infty} \exp(x) = 0$.

It only remains to verify 4. Let $y \in (0, \infty)$. From the earlier properties, there exist x_1 such that $\exp(x_1) < y$ and x_2 such that $\exp(x_2) > y$. Then by the intermediate value theorem, there exists $x \in (x_1, x_2)$ such that $\exp(x) = y$. Thus exp maps onto $(0, \infty)$. It only remains to verify exp is one to one. Suppose then that $x_1 < x_2$. By the mean value theorem, there exists $x \in (x_1, x_2)$ such that

$$\exp(x)(x_2 - x_1) = \exp'(x)(x_2 - x_1) = \exp(x_2) - \exp(x_1).$$

Since $\exp(x) > 0$, it follows $\exp(x_2) \neq \exp(x_1)$. This proves the theorem.

8.3.2 ln And $\log_b$

In this section, everything will be specialized to real valued functions of a real variable.

Definition 8.3.12 ln *is the inverse function of* exp. *Thus* $\ln : (0, \infty) \to \mathbb{R}$, $\ln(\exp(x)) = x$, *and* $\exp(\ln(x)) = x$. *The number* e *is that number such that* $\ln(e) = 1$.

By Corollary 7.10.2, it follows ln is differentiable. This makes possible the following simple theorem.

Theorem 8.3.13 *The following basic properties are available for* ln.

$$\ln'(x) = \frac{1}{x}. \tag{8.15}$$

Also for all $x, y > 0$,

$$\ln(xy) = \ln(x) + \ln(y), \tag{8.16}$$

$$\ln(1) = 0, \ln(x^m) = m \ln(x) \tag{8.17}$$

for all m *an integer.*

Proof: Since $\exp(\ln(x)) = x$ and $\ln'$ exists, it follows

$$x \ln'(x) = \exp(\ln(x)) \ln'(x) = \exp'(\ln(x)) \ln'(x) = 1$$

and this proves 8.15. Next consider 8.16.

$$xy = \exp(\ln(xy)), \ \exp(\ln(x) + \ln(y)) = \exp(\ln(x)) \exp(\ln(y)) = xy.$$

Since exp was shown to be 1-1, it follows $\ln(xy) = \ln(x) + \ln(y)$. Next $\exp(0) = 1$ and $\exp(\ln(1)) = 1$ so $\ln(1) = 0$ again because exp is 1-1. Let

$$f(x) = \ln(x^m) - m \ln(x).$$

$f(1) = \ln(1) - m \ln(1) = 0$. Also, by the chain rule,

$$f'(x) = \frac{1}{x^m} m x^{m-1} - m \frac{1}{x} = 0$$

and so $f(x)$ equals a constant. The constant can only be 0 because $f(1) = 0$. This proves the last formula of 8.17 and completes the proof of the theorem.

The last formula tells how to define x^α for any $x > 0$ and $\alpha \in \mathbb{R}$. I want to stress this is something new. Students are often deceived into thinking they know what x^α means for α a real number. There is no place for such deception in mathematics, however.

Definition 8.3.14 *Define* x^α *for* $x > 0$ *and* $\alpha \in \mathbb{R}$ *by the following formula.*

$$\ln(x^\alpha) = \alpha \ln(x).$$

In other words,

$$x^\alpha \equiv \exp(\alpha \ln(x)).$$

From Theorem 8.3.13 this new definition does not contradict the usual definition in the case where α is an integer.

From this definition, the following properties are obtained.

Proposition 8.3.15 *For $x > 0$ let $f(x) = x^{\alpha}$ where $\alpha \in \mathbb{R}$. Then $f'(x) = \alpha x^{\alpha-1}$. Also $x^{\alpha+\beta} = x^{\alpha}x^{\beta}$ and $(x^{\alpha})^{\beta} = x^{\alpha\beta}$.*

Proof: First consider the claim about the sum of the exponents.

$$\begin{aligned} x^{\alpha+\beta} &\equiv \exp((\alpha+\beta)\ln(x)) = \exp(\alpha\ln(x) + \beta\ln(x)) \\ &= \exp(\alpha\ln(x))\exp(\beta\ln(x)) \equiv x^{\alpha}x^{\beta}. \end{aligned}$$

$$\ln\left((x^{\alpha})^{\beta}\right) = \beta\ln(x^{\alpha}) = \alpha\beta\ln(x), \ \ln\left(x^{\alpha\beta}\right) = \alpha\beta\ln(x).$$

The claim about the derivative follows from the chain rule. $f(x) = \exp(\alpha\ln(x))$ and so

$$f'(x) = \exp(\alpha\ln(x))\frac{\alpha}{x} \equiv \frac{\alpha}{x}x^{\alpha} = \alpha\left(x^{-1}\right)x^{\alpha} = \alpha x^{\alpha-1}.$$

This proves the proposition.

Definition 8.3.16 *Define $\log_b$ for any $b > 0, b \neq 1$ by*

$$\log_b(x) \equiv \frac{\ln(x)}{\ln(b)}.$$

Proposition 8.3.17 *The following hold for $\log_b(x)$.*

1. $b^{\log_b(x)} = x$, $\log_b(b^x) = x$.
2. $\log_b(xy) = \log_b(x) + \log_b(y)$
3. $\log_b(x^{\alpha}) = \alpha\log_b(x)$

Proof:

$$b^{\log_b(x)} \equiv \exp(\ln(b)\log_b(x)) = \exp\left(\ln(b)\frac{\ln(x)}{\ln(b)}\right) = \exp(\ln(x)) = x$$

$$\log_b(b^x) = \frac{\ln(b^x)}{\ln(b)} = \frac{x\ln(b)}{\ln(b)} = x$$

This proves 1.

Now consider 2.

$$\log_b(xy) = \frac{\ln(xy)}{\ln(b)} = \frac{\ln(x)}{\ln(b)} + \frac{\ln(y)}{\ln(b)} = \log_b(x) + \log_b(y).$$

Finally,

$$\log_b(x^{\alpha}) = \frac{\ln(x^{\alpha})}{\ln(b)} = \alpha\frac{\ln(x)}{\ln(b)} = \alpha\log_b(x).$$

8.4 The Binomial Theorem

The following is a very important example known as the binomial series.

Example 8.4.1 *Find a Taylor series for the function $(1+x)^{\alpha}$ centered at 0 valid for $|x| < 1$.*

Use Theorem 8.2.1 to do this. First note that if $y(x) \equiv (1+x)^{\alpha}$, then y is a solution of the following initial value problem.

$$y' - \frac{\alpha}{(1+x)} y = 0, \; y(0) = 1. \tag{8.18}$$

Next it is necessary to observe there is only one solution to this initial value problem. To see this, multiply both sides of the differential equation in 8.18 by $(1+x)^{-\alpha}$. When this is done one obtains

$$\frac{d}{dx}\left((1+x)^{-\alpha} y\right) = (1+x)^{-\alpha}\left(y' - \frac{\alpha}{(1+x)} y\right) = 0. \tag{8.19}$$

Therefore, from 8.19, there must exist a constant, C, such that

$$(1+x)^{-\alpha} y = C.$$

However, $y(0) = 1$ and so it must be that $C = 1$. Therefore, there is exactly one solution to the initial value problem in 8.18 and it is $y(x) = (1+x)^{\alpha}$.

The strategy for finding the Taylor series of this function consists of finding a series which solves the initial value problem above. Let

$$y(x) \equiv \sum_{n=0}^{\infty} a_n x^n \tag{8.20}$$

be a solution to 8.18. Of course it is not known at this time whether such a series exists. However, the process of finding it will demonstrate its existence. From Theorem 8.2.1 and the initial value problem,

$$(1+x)\sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} \alpha a_n x^n = 0$$

and so

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n (n-\alpha) x^n = 0$$

Changing the order variable of summation in the first sum,

$$\sum_{n=0}^{\infty} a_{n+1}(n+1) x^n + \sum_{n=0}^{\infty} a_n (n-\alpha) x^n = 0$$

and from Corollary 8.2.2 and the initial condition for 8.18 this requires

$$a_{n+1} = \frac{a_n(\alpha - n)}{n+1}, a_0 = 1. \tag{8.21}$$

Therefore, from 8.21 and letting $n = 0$, $a_1 = \alpha$, then using 8.21 again along with this information,

$$a_2 = \frac{\alpha(\alpha-1)}{2}.$$

Using the same process,

$$a_3 = \frac{\left(\frac{\alpha(\alpha-1)}{2}\right)(\alpha-2)}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{3!}.$$

By now you can spot the pattern. In general,

$$a_n = \frac{\overbrace{\alpha\left(\alpha-1\right)\cdots\left(\alpha-n+1\right)}^{n \text{ of these factors}}}{n!}.$$

Therefore, the candidate for the Taylor series is

$$y\left(x\right) = \sum_{n=0}^{\infty} \frac{\alpha\left(\alpha-1\right)\cdots\left(\alpha-n+1\right)}{n!} x^n.$$

Furthermore, the above discussion shows this series solves the initial value problem on its interval of convergence. It only remains to show the radius of convergence of this series equals 1. It will then follow that this series equals $(1+x)^\alpha$ because of uniqueness of the initial value problem. To find the radius of convergence, use the ratio test. Thus the ratio of the absolute values of $(n+1)^{st}$ term to the absolute value of the n^{th} term is

$$\frac{\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{(n+1)n!}\right| |x|^{n+1}}{\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\right| |x|^n} = |x| \frac{|\alpha-n|}{n+1} \to |x|$$

showing that the radius of convergence is 1 since the series converges if $|x| < 1$ and diverges if $|x| > 1$.

The expression, $\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ is often denoted as $\binom{\alpha}{n}$. With this notation, the following theorem has been established.

Theorem 8.4.2 *Let α be a real number and let $|x| < 1$. Then*

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

There is a very interesting issue related to the above theorem which illustrates the limitation of power series. The function $f(x) = (1+x)^\alpha$ makes sense for all $x > -1$ but one is only able to describe it with a power series on the interval $(-1,1)$. Think about this. The above technique is a standard one for obtaining solutions of differential equations and this example illustrates a deficiency in the method.

To completely understand power series, it is necessary to take a course in complex analysis. It turns out that the right way to consider Taylor series is through the use of geometric series and something called the Cauchy integral formula of complex analysis. However, these are topics for another course.

8.5 Exercises

1. In each of the following, assume the relation defines y as a function of x for values of x and y of interest and find $y'(x)$.

   (a) $xy^2 + \sin(y) = x^3 + 1$

   (b) $y^3 + x\cos\left(y^2\right) = x^4$

   (c) $y\cos(x) = \tan(y)\cos\left(x^2\right) + 2$

   (d) $\left(x^2+y^2\right)^6 = x^3y + 3$

(e) $\frac{xy^2+y}{y^5+x} + \cos(y) = 7$

(f) $\sqrt{x^2 + y^4} \sin(y) = 3x$

(g) $y^3 \sin(x) + y^2 x^2 = 2^{x^2} y + \ln|y|$

(h) $y^2 \sin(y) x + \log_3(xy) = y^2 + 11$

(i) $\sin\left(x^2 + y^2\right) + \sec(xy) = e^{x+y} + y2^y + 2$

(j) $\sin\left(\tan\left(xy^2\right)\right) + y^3 = 16$

(k) $\cos(\sec(\tan(y))) + \ln(5 + \sin(xy)) = x^2 y + 3$

2. In each of the following, assume the relation defines y as a function of x for values of x and y of interest. Use the chain rule to show y satisfies the given differential equation.

   (a) $x^2 y + \sin y = 7,\ \left(x^2 + \cos y\right) y' + 2xy = 0.$

   (b) $x^2 y^3 + \sin\left(y^2\right) = 5,\ 2xy^3 + \left(3x^2y^2 + 2\left(\cos\left(y^2\right)\right) y\right) y' = 0.$

   (c) $y^2 \sin(y) + xy = 6,$

$$\left(2y(\sin(y)) + y^2(\cos(y)) + x\right) y' + y = 0.$$

3. Show that if $D(g) \subseteq U \subseteq D(f)$, and if f and g are both one to one, then $f \circ g$ is also one to one.

4. The number e is that number such that $\ln e = 1$. Prove $e^x = \exp(x)$.

5. Find a formula for $\frac{dy}{dx}$ for $y = b^x$. Prove your formula.

6. Let $y = x^x$ for $x \in \mathbb{R}$. Find $y'(x)$.

7. The logarithm test states the following. Suppose $a_k \neq 0$ for large k and that $p = \lim_{k\to\infty} \frac{\ln\left(\frac{1}{|a_k|}\right)}{\ln k}$ exists. If $p > 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely. If $p < 1$, then the series, $\sum_{k=1}^{\infty} a_k$ does not converge absolutely. Prove this theorem.

8. Suppose $f(x+y) = f(x) + f(y)$ and f is continuous at 0. Find all solutions to this functional equation which are continuous at $x = 0$. Now find all solutions which are bounded near 0. Next if you want an even more interesting version of this, find all solutions whose graphs are not dense in the plane. (A set S is dense in the plane if for every $(a, b) \in \mathbb{R} \times \mathbb{R}$ and $r > 0$, there exists $(x, y) \in S$ such that

$$\sqrt{(x-a)^2 + (y-b)^2} < r.)$$

   This is called the Cauchy equation.

9. Suppose $f(x+y) = f(x) f(y)$ and f is continuous and not identically zero. Find all solutions to this functional equation. **Hint:** First show the functional equation requires $f > 0$.

10. Suppose $f(xy) = f(x) + f(y)$ for $x, y > 0$. Suppose also f is continuous. Find all solutions to this functional equation.

11. Using the Cauchy condensation test, determine the convergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$. Now determine the convergence of $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^{1.001}}$.

12. Find the values of p for which the following series converges and the values of p for which it diverges.
$$\sum_{k=4}^{\infty} \frac{1}{\ln^p (\ln (k)) \ln (k) k}$$

13. For p a positive number, determine the convergence of
$$\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$$
for various values of p.

14. Determine whether the following series converge absolutely, conditionally, or not at all and give reasons for your answers.

    (a) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(k^5)}{k}$
    (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(k^5)}{k^{1.01}}$
    (c) $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(1.01)^n}$
    (d) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$
    (e) $\sum_{n=1}^{\infty} (-1)^n \tan\left(\frac{1}{n^2}\right)$
    (f) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$
    (g) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\sqrt{n}}{n^2+1}\right)$

15. Find the values of x such that the series $\sum_{k=1}^{\infty} \frac{1-e^x}{k}$ converges.

16. De Moivre's theorem says
$$[r(\cos t + i \sin t)]^n = r^n (\cos nt + i \sin nt)$$
for n a positive integer. Prove this formula by induction. Does this formula continue to hold for all integers, n, even negative integers? Explain.

17. Using De Moivre's theorem, show that if $z \in \mathbb{C}$ then z has n distinct n^{th} roots. **Hint:**Letting $z = x + iy$,
$$z = |z| \left(\frac{x}{|z|} + i \frac{y}{|z|} \right)$$
and argue $\left(\frac{x}{|z|}, \frac{y}{|z|}\right)$ is a point on the unit circle. Hence $z = |z| (\cos(\theta) + i \sin(\theta))$. Then
$$w = |w| (\cos(\alpha) + i \sin(\alpha))$$
is an n^{th} root if and only if $(|w| (\cos(\alpha) + i \sin(\alpha)))^n = z$. Show this happens exactly when $|w| = \sqrt[n]{|z|}$ and $\alpha = \frac{\theta + 2k\pi}{n}$ for $k = 0, 1, \cdots, n$.

18. Using De Moivre's theorem from Problem 16, derive a formula for $\sin(5x)$ and one for $\cos(5x)$.

19. Suppose $\sum_{n=0}^{\infty} a_n (x-c)^n$ is a power series with radius of convergence r. Show the series converge uniformly on any interval $[a, b]$ where $[a, b] \subseteq (c - r, c + r)$.

20. Find the disc of convergence of the series $\sum \frac{x^n}{n^p}$ for various values of p. **Hint:** Use Dirichlet's test and De Moivre's theorem.

21. Show
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
for all $x \in \mathbb{R}$ where e is the number such that $\ln e = 1$. Thus
$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$
Show e is irrational. **Hint:** If $e = p/q$ for p, q positive integers, then argue
$$q!\left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}\right)$$
is an integer. However, you can also show
$$q!\left(\sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{q} \frac{1}{k!}\right) < 1$$

8.6 L'Hôpital's Rule

There is an interesting rule which is often useful for evaluating difficult limits called L'Hôpital's[1] rule. The best versions of this rule are based on the Cauchy Mean value theorem, Theorem 7.8.2 on Page 132.

Theorem 8.6.1 *Let $[a, b] \subseteq [-\infty, \infty]$ and suppose f, g are functions which satisfy,*
$$\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0, \tag{8.22}$$
and f' and g' exist on (a, b) with $g'(x) \neq 0$ on (a, b). Suppose also that
$$\lim_{x \to b-} \frac{f'(x)}{g'(x)} = L. \tag{8.23}$$
Then
$$\lim_{x \to b-} \frac{f(x)}{g(x)} = L. \tag{8.24}$$

Proof: By the definition of limit and 8.23 there exists $c < b$ such that if $t > c$, then
$$\left|\frac{f'(t)}{g'(t)} - L\right| < \frac{\varepsilon}{2}.$$
Now pick x, y such that $c < x < y < b$. By the Cauchy mean value theorem, there exists $t \in (x, y)$ such that
$$g'(t)(f(x) - f(y)) = f'(t)(g(x) - g(y)).$$

[1] L'Hôpital published the first calculus book in 1696. This rule, named after him, appeared in this book. The rule was actually due to Bernoulli who had been L'Hôpital's teacher. L'Hôpital did not claim the rule as his own but Bernoulli accused him of plagarism. Nevertheless, this rule has become known as L'Hôpital's rule ever since. The version of the rule presented here is superior to what was discovered by Bernoulli and depends on the Cauchy mean value theorem which was found over 100 years after the time of L'Hôpital.

Since $g'(s) \neq 0$ for all $s \in (a,b)$ it follows $g(x) - g(y) \neq 0$. Therefore,

$$\frac{f'(t)}{g'(t)} = \frac{f(x) - f(y)}{g(x) - g(y)}$$

and so, since $t > c$,

$$\left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| < \frac{\varepsilon}{2}.$$

Now letting $y \to b-$,

$$\left| \frac{f(x)}{g(x)} - L \right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows 8.24.

The following corollary is proved in the same way.

Corollary 8.6.2 *Let $[a,b] \subseteq [-\infty, \infty]$ and suppose f, g are functions which satisfy,*

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0, \tag{8.25}$$

and f' and g' exist on (a,b) with $g'(x) \neq 0$ on (a,b). Suppose also that

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L. \tag{8.26}$$

Then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L. \tag{8.27}$$

Here is a simple example which illustrates the use of this rule.

Example 8.6.3 *Find* $\lim_{x \to 0} \frac{5x + \sin 3x}{\tan 7x}$.

The conditions of L'Hôpital's rule are satisfied because the numerator and denominator both converge to 0 and the derivative of the denominator is nonzero for x close to 0. Therefore, if the limit of the quotient of the derivatives exists, it will equal the limit of the original function. Thus,

$$\lim_{x \to 0} \frac{5x + \sin 3x}{\tan 7x} = \lim_{x \to 0} \frac{5 + 3\cos 3x}{7 \sec^2(7x)} = \frac{8}{7}.$$

Sometimes you have to use L'Hôpital's rule more than once.

Example 8.6.4 *Find* $\lim_{x \to 0} \frac{\sin x - x}{x^3}$.

Note that $\lim_{x \to 0} (\sin x - x) = 0$ and $\lim_{x \to 0} x^3 = 0$. Also, the derivative of the denominator is nonzero for x close to 0. Therefore, if $\lim_{x \to 0} \frac{\cos x - 1}{3x^2}$ exists and equals L, it will follow from L'Hôpital's rule that the original limit exists and equals L. However, $\lim_{x \to 0} (\cos x - 1) = 0$ and $\lim_{x \to 0} 3x^2 = 0$ so L'Hôpital's rule can be applied again to consider $\lim_{x \to 0} \frac{-\sin x}{6x}$. From L'Hôpital's rule, if this limit exists and equals L, it will follow that $\lim_{x \to 0} \frac{\cos x - 1}{3x^2} = L$ and consequently $\lim_{x \to 0} \frac{\sin x - x}{x^3} = L$. But from Proposition 8.3.6, $\lim_{x \to 0} \frac{-\sin x}{6x} = \frac{-1}{6}$. Therefore, by L'Hôpital's rule, $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{-1}{6}$.

Warning 8.6.5 ***Be sure to check the assumptions of L'Hôpital's rule before using it.***

Example 8.6.6 *Find* $\lim_{x\to 0+} \frac{\cos 2x}{x}$.

The numerator becomes close to 1 and the denominator gets close to 0. Therefore, the assumptions of L'Hôpital's rule do not hold and so it does not apply. In fact there is no limit unless you define the limit to equal $+\infty$. Now lets try to use the conclusion of L'Hôpital's rule even though the conditions for using this rule are not verified. Take the derivative of the numerator and the denominator which yields $\frac{-2\sin 2x}{1}$, an expression whose limit as $x \to 0+$ equals 0. This is a good illustration of the above warning.

Some people get the unfortunate idea that one can find limits by doing experiments with a calculator. If the limit is taken as x gets close to 0, these people think one can find the limit by evaluating the function at values of x which are closer and closer to 0. Theoretically, this should work although you have no way of knowing how small you need to take x to get a good estimate of the limit. In practice, the procedure may fail miserably.

Example 8.6.7 *Find* $\lim_{x\to 0} \frac{\ln\left|1+x^{10}\right|}{x^{10}}$.

This limit equals $\lim_{y\to 0} \frac{\ln|1+y|}{y} = \lim_{y\to 0} \frac{\left(\frac{1}{1+y}\right)}{1} = 1$ where L'Hôpital's rule has been used. This is an amusing example. You should plug .001 in to the function, $\frac{\ln\left|1+x^{10}\right|}{x^{10}}$ and see what your calculator or computer gives you. If it is like mine, it will give the answer, 0 and will keep on returning the answer of 0 for smaller numbers than .001. This illustrates the folly of trying to compute limits through calculator or computer experiments. Indeed, you could say that a calculator is as useful for taking limits as a bicycle is for swimming.

There is another form of L'Hôpital's rule in which $\lim_{x\to b-} f(x) = \pm\infty$ and $\lim_{x\to b-} g(x) = \pm\infty$.

Theorem 8.6.8 *Let* $[a,b] \subseteq [-\infty,\infty]$ *and suppose* f, g *are functions which satisfy,*

$$\lim_{x\to b-} f(x) = \pm\infty \text{ and } \lim_{x\to b-} g(x) = \pm\infty, \tag{8.28}$$

and f' *and* g' *exist on* (a,b) *with* $g'(x) \neq 0$ *on* (a,b). *Suppose also*

$$\lim_{x\to b-} \frac{f'(x)}{g'(x)} = L. \tag{8.29}$$

Then

$$\lim_{x\to b-} \frac{f(x)}{g(x)} = L. \tag{8.30}$$

Proof: By the definition of limit and 8.29 there exists $c < b$ such that if $t > c$, then

$$\left|\frac{f'(t)}{g'(t)} - L\right| < \frac{\varepsilon}{2}.$$

Now pick x, y such that $c < x < y < b$. By the Cauchy mean value theorem, there exists $t \in (x,y)$ such that

$$g'(t)(f(x) - f(y)) = f'(t)(g(x) - g(y)).$$

Since $g'(s) \neq 0$ on (a,b), it follows $g(x) - g(y) \neq 0$. Therefore,

$$\frac{f'(t)}{g'(t)} = \frac{f(x) - f(y)}{g(x) - g(y)}$$

and so, since $t > c$,

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - L\right| < \frac{\varepsilon}{2}.$$

Now this implies

$$\left| \frac{f(y)}{g(y)} \frac{\left(\frac{f(x)}{f(y)} - 1\right)}{\left(\frac{g(x)}{g(y)} - 1\right)} - L \right| < \frac{\varepsilon}{2}$$

where for all y large enough, both $\frac{f(x)}{f(y)} - 1$ and $\frac{g(x)}{g(y)} - 1$ are not equal to zero. Continuing to rewrite the above inequality yields

$$\left| \frac{f(y)}{g(y)} - L \frac{\left(\frac{g(x)}{g(y)} - 1\right)}{\left(\frac{f(x)}{f(y)} - 1\right)} \right| < \frac{\varepsilon}{2} \left| \frac{\left(\frac{g(x)}{g(y)} - 1\right)}{\left(\frac{f(x)}{f(y)} - 1\right)} \right|.$$

Therefore, for y large enough,

$$\left| \frac{f(y)}{g(y)} - L \right| \leq \left| L - L \frac{\left(\frac{g(x)}{g(y)} - 1\right)}{\left(\frac{f(x)}{f(y)} - 1\right)} \right| + \frac{\varepsilon}{2} \left| \frac{\left(\frac{g(x)}{g(y)} - 1\right)}{\left(\frac{f(x)}{f(y)} - 1\right)} \right| < \varepsilon$$

due to the assumption 8.28 which implies

$$\lim_{y \to b-} \frac{\left(\frac{g(x)}{g(y)} - 1\right)}{\left(\frac{f(x)}{f(y)} - 1\right)} = 1.$$

Therefore, whenever y is large enough,

$$\left| \frac{f(y)}{g(y)} - L \right| < \varepsilon$$

and this is what is meant by 8.30. This proves the theorem.

As before, there is no essential difference between the proof in the case where $x \to b-$ and the proof when $x \to a+$. This observation is stated as the next corollary.

Corollary 8.6.9 *Let $[a, b] \subseteq [-\infty, \infty]$ and suppose f, g are functions which satisfy,*

$$\lim_{x \to a+} f(x) = \pm\infty \text{ and } \lim_{x \to a+} g(x) = \pm\infty, \tag{8.31}$$

and f' and g' exist on (a, b) with $g'(x) \neq 0$ on (a, b). Suppose also that

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L. \tag{8.32}$$

Then

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L. \tag{8.33}$$

Theorems 8.6.1 8.6.8 and Corollaries 8.6.2 and 8.6.9 will be referred to as L'Hôpital's rule from now on. Theorem 8.6.1 and Corollary 8.6.2 involve the notion of indeterminate forms of the form $\frac{0}{0}$. Please do not think any meaning is being assigned to the nonsense expression $\frac{0}{0}$. It is just a symbol to help remember the sort of thing described by Theorem 8.6.1 and Corollary 8.6.2. Theorem 8.6.8 and Corollary 8.6.9 deal with indeterminate forms which are of the form $\frac{\pm\infty}{\infty}$. Again, this is just a symbol which is helpful in remembering the sort of thing being considered. There are other indeterminate forms which can be reduced to these forms just discussed. Don't ever try to assign meaning to such symbols.

Example 8.6.10 *Find* $\lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y$.

It is good to first see why this is called an indeterminate form. One might think that as $y\to\infty$, it follows $x/y\to 0$ and so $1+\frac{x}{y}\to 1$. Now 1 raised to anything is 1 and so it would seem this limit should equal 1. On the other hand, if $x>0$, $1+\frac{x}{y}>1$ and a number raised to higher and higher powers should approach ∞. It really isn't clear what this limit should be. It is an indeterminate form which can be described as 1^∞. By definition,

$$\left(1+\frac{x}{y}\right)^y = \exp\left(y\ln\left(1+\frac{x}{y}\right)\right).$$

Now using L'Hôpital's rule,

$$\begin{aligned}
\lim_{y\to\infty} y\ln\left(1+\frac{x}{y}\right) &= \lim_{y\to\infty}\frac{\ln\left(1+\frac{x}{y}\right)}{1/y}\\
&= \lim_{y\to\infty}\frac{\frac{1}{1+(x/y)}\left(-x/y^2\right)}{\left(-1/y^2\right)}\\
&= \lim_{y\to\infty}\frac{x}{1+(x/y)} = x
\end{aligned}$$

Therefore,

$$\lim_{y\to\infty} y\ln\left(1+\frac{x}{y}\right) = x$$

Since exp is continuous, it follows

$$\lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y = \lim_{y\to\infty}\exp\left(y\ln\left(1+\frac{x}{y}\right)\right) = e^x.$$

8.6.1 Interest Compounded Continuously

Suppose you put money in the bank and it accrues interest at the rate of r per payment period. These terms need a little explanation. If the payment period is one month, and you started with \$100 then the amount at the end of one month would equal $100(1+r) = 100+100r$. In this the second term is the interest and the first is called the principal. Now you have $100(1+r)$ in the bank. This becomes the new principal. How much will you have at the end of the second month? By analogy to what was just done it would equal

$$100(1+r)+100(1+r)r = 100(1+r)^2.$$

In general, the amount you would have at the end of n months is $100(1+r)^n$.

When a bank says they offer 6% compounded monthly, this means r, the rate per payment period equals .06/12. Consider the problem of a rate of r per year and compounding the interest n times a year and letting n increase without bound. This is what is meant by compounding continuously. The interest rate per payment period is then r/n and the number of payment periods after time t years is approximately tn. From the above the amount in the account after t years is

$$P\left(1+\frac{r}{n}\right)^{nt} \tag{8.34}$$

Recall from Example 8.6.10 that $\lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y = e^x$. The expression in 8.34 can be written as

$$P\left[\left(1+\frac{r}{n}\right)^n\right]^t$$

and so, taking the limit as $n \to \infty$, you get

$$Pe^{rt} = A.$$

This shows how to compound interest continuously.

Example 8.6.11 *Suppose you have $100 and you put it in a savings account which pays 6% compounded continuously. How much will you have at the end of 4 years?*

From the above discussion, this would be $100e^{(.06)4} = 127.12$. Thus, in 4 years, you would gain interest of about $27.

8.7 Exercises

1. Find the limits.

   (a) $\lim_{x\to 0} \frac{3x-4\sin 3x}{\tan 3x}$

   (b) $\lim_{x\to \frac{\pi}{2}-} (\tan x)^{x-(\pi/2)}$

   (c) $\lim_{x\to 1} \frac{\arctan(4x-4)}{\arcsin(4x-4)}$

   (d) $\lim_{x\to 0} \frac{\arctan 3x - 3x}{x^3}$

   (e) $\lim_{x\to 0+} \frac{9^{\sec x - 1}-1}{3^{\sec x - 1}-1}$

   (f) $\lim_{x\to 0} \frac{3x+\sin 4x}{\tan 2x}$

   (g) $\lim_{x\to \pi/2} \frac{\ln(\sin x)}{x-(\pi/2)}$

   (h) $\lim_{x\to 0} \frac{\cosh 2x - 1}{x^2}$

   (i) $\lim_{x\to 0} \frac{-\arctan x + x}{x^3}$

   (j) $\lim_{x\to 0} \frac{x^8 \sin \frac{1}{x}}{\sin 3x}$

   (k) $\lim_{x\to \infty} (1+5^x)^{\frac{2}{x}}$

   (l) $\lim_{x\to 0} \frac{-2x+3\sin x}{x}$

   (m) $\lim_{x\to 1} \frac{\ln(\cos(x-1))}{(x-1)^2}$

   (n) $\lim_{x\to 0+} \sin^{\frac{1}{x}} x$

   (o) $\lim_{x\to 0} (\csc 5x - \cot 5x)$

   (p) $\lim_{x\to 0+} \frac{3^{\sin x}-1}{2^{\sin x}-1}$

   (q) $\lim_{x\to 0+} (4x)^{x^2}$

   (r) $\lim_{x\to \infty} \frac{x^{10}}{(1.01)^x}$

   (s) $\lim_{x\to 0} (\cos 4x)^{(1/x^2)}$

2. Find the following limits.

   (a) $\lim_{x\to 0+} \frac{1-\sqrt{\cos 2x}}{\sin^4(4\sqrt{x})}$.

   (b) $\lim_{x\to 0} \frac{2^{x^2}-2^{5x}}{\sin\left(\frac{x^2}{5}\right)-\sin(3x)}$.

(c) $\lim_{n\to\infty} n\left(\sqrt[n]{7}-1\right)$.

(d) $\lim_{x\to\infty}\left(\frac{3x+2}{5x-9}\right)^{x^2}$.

(e) $\lim_{x\to\infty}\left(\frac{3x+2}{5x-9}\right)^{1/x}$.

(f) $\lim_{n\to\infty}\left(\cos\frac{2x}{\sqrt{n}}\right)^{n}$.

(g) $\lim_{n\to\infty}\left(\cos\frac{2x}{\sqrt{5n}}\right)^{n}$.

(h) $\lim_{x\to 3}\frac{x^x-27}{x-3}$.

(i) $\lim_{n\to\infty}\cos\left(\pi\frac{\sqrt{4n^2+13n}}{n}\right)$.

(j) $\lim_{x\to\infty}\left(\sqrt[3]{x^3+7x^2}-\sqrt{x^2-11x}\right)$.

(k) $\lim_{x\to\infty}\left(\sqrt[5]{x^5+7x^4}-\sqrt[3]{x^3-11x^2}\right)$.

(l) $\lim_{x\to\infty}\left(\frac{5x^2+7}{2x^2-11}\right)^{\frac{x}{1-x}}$.

(m) $\lim_{x\to\infty}\left(\frac{5x^2+7}{2x^2-11}\right)^{\frac{x\ln x}{1-x}}$.

(n) $\lim_{x\to 0+}\frac{\ln\left(e^{2x^2}+7\sqrt{x}\right)}{\sinh\left(\sqrt{x}\right)}$.

(o) $\lim_{x\to 0+}\frac{\sqrt[7]{x}-\sqrt[5]{x}}{\sqrt[9]{x}-\sqrt[11]{x}}$.

3. Find the following limits.

(a) $\lim_{x\to 0+}(1+3x)^{\cot 2x}$

(b) $\lim_{x\to 0}\frac{\sin x-x}{x^2}=0$

(c) $\lim_{x\to 0}\frac{\sin x-x}{x^3}$

(d) $\lim_{x\to 0}\frac{\tan(\sin x)-\sin(\tan x)}{x^7}$

(e) $\lim_{x\to 0}\frac{\tan(\sin 2x)-\sin(\tan 2x)}{x^7}$

(f) $\lim_{x\to 0}\frac{\sin\left(x^2\right)-\sin^2(x)}{x^4}$

(g) $\lim_{x\to 0}\frac{e^{-\left(1/x^2\right)}}{x}$

(h) $\lim_{x\to 0}\left(\frac{1}{x}-\cot(x)\right)$

(i) $\lim_{x\to 0}\frac{\cos(\sin x)-1}{x^2}$

(j) $\lim_{x\to\infty}\left(x^2\left(4x^4+7\right)^{1/2}-2x^4\right)$

(k) $\lim_{x\to 0}\frac{\cos(x)-\cos(4x)}{\tan(x^2)}$

(l) $\lim_{x\to 0}\frac{\arctan(3x)}{x}$

(m) $\lim_{x\to\infty}\left[\left(x^9+5x^6\right)^{1/3}-x^3\right]$

4. Suppose you want to have \$2000 saved at the end of 5 years. How much money should you place into an account which pays 7% per year compounded continuously?

5. Using a good calculator, find $e^{.06}-\left(1+\frac{.06}{360}\right)^{360}$. Explain why this gives a measure of the difference between compounding continuously and compounding daily.

8.8 Multiplication Of Power Series

Next consider the problem of multiplying two power series.

Theorem 8.8.1 *Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ and $\sum_{n=0}^{\infty} b_n (x-a)^n$ be two power series having radii of convergence r_1 and r_2, both positive. Then*

$$\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n (x-a)^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-a)^n$$

whenever $|x-a| < r \equiv \min(r_1, r_2)$.

Proof: By Theorem 8.1.3 both series converge absolutely if $|x-a| < r$. Therefore, by Theorem 5.4.8

$$\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n (x-a)^n\right) =$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k (x-a)^k b_{n-k} (x-a)^{n-k} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-a)^n .$$

This proves the theorem.

The significance of this theorem in terms of applications is that it states you can multiply power series just as you would multiply polynomials and everything will be all right on the common interval of convergence.

This theorem can be used to find Taylor series which would perhaps be hard to find without it. Here is an example.

Example 8.8.2 *Find the Taylor series for* $e^x \sin x$ *centered at* $x = 0$.

All that is required is to multiply

$$\left(\overbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\cdots}^{e^x}\right)\left(\overbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}^{\sin x}\right)$$

From the above theorem the result should be

$$x + x^2 + \left(-\frac{1}{3!} + \frac{1}{2!}\right)x^3 + \cdots$$

$$= x + x^2 + \frac{1}{3}x^3 + \cdots$$

You can continue this way and get the following to a few more terms.

$$x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 - \frac{1}{630}x^7 + \cdots$$

I don't see a pattern in these coefficients but I can go on generating them as long as I want. (In practice this tends to not be very long.) I also know the resulting power series will converge for all x because both the series for e^x and the one for $\sin x$ converge for all x.

Example 8.8.3 *Find the Taylor series for* $\tan x$ *centered at* $x = 0$.

Lets suppose it has a Taylor series $a_0 + a_1 x + a_2 x^2 + \cdots$. Then

$$\left(a_0 + a_1 x + a_2 x^2 + \cdots\right)\left(\overbrace{1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots}^{\cos x}\right) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right).$$

Using the above, $a_0 = 0, a_1 x = x$ so $a_1 = 1$, $\left(0\left(\frac{-1}{2}\right) + a_2\right)x^2 = 0$ so $a_2 = 0$. $\left(a_3 - \frac{a_1}{2}\right)x^3 = \frac{-1}{3!}x^3$ so $a_3 - \frac{1}{2} = -\frac{1}{6}$ so $a_3 = \frac{1}{3}$. Clearly one can continue in this manner. Thus the first several terms of the power series for tan are

$$\tan x = x + \frac{1}{3}x^3 + \cdots.$$

You can go on calculating these terms and find the next two yielding

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

This is a very significant technique because, as you see, there does not appear to be a very simple pattern for the coefficients of the power series for $\tan x$. Of course there are some issues here about whether $\tan x$ even has a power series, but if it does, the above must be it. In fact, $\tan(x)$ will have a power series valid on some interval centered at 0 and this becomes completely obvious when one uses methods from complex analysis but it isn't too obvious at this point. If you are interested in this issue, read the last section of the chapter. Note also that what has been accomplished is to divide the power series for $\sin x$ by the power series for $\cos x$ just like they were polynomials.

8.9 Exercises

1. Find the radius of convergence of the following.
   (a) $\sum_{k=1}^{\infty}\left(\frac{x}{2}\right)^n$
   (b) $\sum_{k=1}^{\infty}\sin\left(\frac{1}{n}\right)3^n x^n$
   (c) $\sum_{k=0}^{\infty} k! x^k$
   (d) $\sum_{n=0}^{\infty}\frac{(3n)^n}{(3n)!}x^n$
   (e) $\sum_{n=0}^{\infty}\frac{(2n)^n}{(2n)!}x^n$
2. Find $\sum_{k=1}^{\infty} k2^{-k}$.
3. Find $\sum_{k=1}^{\infty} k^2 3^{-k}$.
4. Find $\sum_{k=1}^{\infty}\frac{2^{-k}}{k}$.
5. Find $\sum_{k=1}^{\infty}\frac{3^{-k}}{k}$.
6. Find where the series $\sum_{k=1}^{\infty}\frac{1-e^x}{k}$ converges. Then find a formula for the infinite sum.
7. Show there exists a function f which is continuous on $[0,1]$ and an infinite series of the form $\sum_{k=1}^{\infty} p_k(x)$ where each p_k is a polynomial which converges uniformly to $f(x)$ on $[0,1]$. Thus it makes absolutely no sense to write something like $f'(x) = \sum_{k=1}^{\infty} p_k'(x)$. **Hint:** Use the Weierstrass approximation theorem.

8. Suppose $f(x+y) = f(x) f(y)$ and f is continuous. Find all possible solutions to the functional equation for f where $f(x) > 0$ and f is continuous.

9. For $x > 0$, suppose $f(xy) = f(x) + f(y)$ and f is continuous. Find all possible solutions to this functional equation which are differentiable.

10. Find the power series centered at 0 for the function $1/\left(1+x^2\right)$ and give the radius of convergence. Where does the function make sense? Where does the power series equal the function?

11. Find a power series for the function, $f(x) \equiv \frac{\sin(\sqrt{x})}{\sqrt{x}}$ for $x > 0$. Where does $f(x)$ make sense? Where does the power series you found converge?

12. Use the power series technique which was applied in Example 8.4.1 to consider the initial value problem $y' = y, y(0) = 1$. This yields another way to obtain the power series for e^x.

13. Use the power series technique on the initial value problem $y' + y = 0,\ y(0) = 1$. What is the solution to this initial value problem?

14. Use the power series technique to find solutions in terms of power series to the initial value problem
$$y'' + xy = 0,\ y(0) = 0, y'(0) = 1.$$
Tell where your solution gives a valid description of a solution for the initial value problem. **Hint:** This is a little different but you proceed the same way as in Example 8.4.1. The main difference is you have to do two differentiations of the power series instead of one.

15. Find several terms of a power series solution to the nonlinear initial value problem
$$y'' + a\sin(y) = 0,\ y(0) = 1, y'(0) = 0.$$
This is the equation which governs the vibration of a pendulum. Explain why there exists a power series which gives the solution to the above initial value problem. It might be useful to use the formula of Problem 10 on Page 134. Multiply the equation by y' and identify what you have obtained as the derivativre of an interesting quantity which must be constant constant.

16. Suppose the function, e^x is defined in terms of a power series, $e^x \equiv \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Use Theorem 5.4.8 on Page 84 to show directly the usual law of exponents,
$$e^{x+y} = e^x e^y.$$
Be sure to check all the hypotheses.

17. Define the following function[2]:
$$f(x) \equiv \begin{cases} e^{-\left(1/x^2\right)} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$

[2]Surprisingly, this function is very important to those who use modern techniques to study differential equations. One needs to consider test functions which have the property they have infinitely many derivatives but vanish outside of some interval. The theory of complex variables can be used to show there are no examples of such functions if they have a valid power series expansion. It even becomes a little questionable whether such strange functions even exist at all. Nevertheless, they do, there are enough of them, and it is this very example which is used to show this.

Show that $f^{(k)}(x)$ exists for all k and for all x. Show also that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Therefore, the power series for $f(x)$ is of the form $\sum_{k=0}^{\infty} 0x^k$ and it converges for all values of x. However, it fails to converge to $f(x)$ except at the single point, $x = 0$.

18. Let $f_n(x) \equiv \left(\frac{1}{n} + x^2\right)^{1/2}$. Show that for all x,
$$||x| - f_n(x)| \leq \frac{1}{\sqrt{n}}.$$
Now show $f_n'(0) = 0$ for all n and so $f_n'(0) \to 0$. However, the function, $f(x) \equiv |x|$ has no derivative at $x = 0$. Thus even though $f_n(x) \to f(x)$ for all x, you cannot say that $f_n'(0) \to f'(0)$.

19. Let the functions, $f_n(x)$ be given in Problem 18 and consider
$$g_1(x) = f_1(x),\ g_n(x) = f_n(x) - f_{n-1}(x) \text{ if } n > 1.$$
Show that for all x,
$$\sum_{k=0}^{\infty} g_k(x) = |x|$$
and that $g_k'(0) = 0$ for all k. Therefore, you can't differentiate the series term by term and get the right answer[3].

20. Use the theorem about the binomial series to give a proof of the binomial theorem
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
whenever n is a positive integer.

21. Find the power series for $\sin\left(x^2\right)$ by plugging in x^2 where ever there is an x in the power series for $\sin x$. How do you know this is the power series for $\sin\left(x^2\right)$?

22. Find the first several terms of the power series for $\sin^2(x)$ by multiplying the power series for $\sin(x)$. Next use the trig. identity, $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and the power series for $\cos(2x)$ to find the power series.

23. Find the power series for $f(x) = \frac{1}{\sqrt{1-x^2}}$.

24. Let a, b be two positive numbers and let $p > 1$. Choose q such that
$$\frac{1}{p} + \frac{1}{q} = 1.$$
Now verify the important inequality
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$
Hint: You might try considering $f(a) = \frac{a^p}{p} + \frac{b^q}{q} - ab$ for fixed $b > 0$ and examine its graph using the derivative.

[3]How bad can this get? It can be much worse than this. In fact, there are functions which are continuous everywhere and differentiable nowhere. We typically don't have names for them but they are there just the same. Every such function can be written as an infinite sum of polynomials which of course have derivatives at every point. Thus it is nonsense to differentiate an infinite sum term by term without a theorem of some sort.

8.10 The Fundamental Theorem Of Algebra

The fundamental theorem of algebra states that every non constant polynomial having coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$. If $\mathbb{C}$ is replaced by $\mathbb{R}$, this is not true because of the example, $x^2+1=0$. This theorem is a very remarkable result and notwithstanding its title, all the best proofs of it depend on either analysis or topology. It was first proved by Gauss in 1797. The proof given here follows Rudin [30]. See also Hardy [18] for a similar proof, more discussion and references. The best proof is found in the theory of complex analysis. Recall De Moivre's theorem, Problem 16 on Page 159 from trigonometry which is listed here for convenience.

Theorem 8.10.1 *Let $r>0$ be given. Then if n is a positive integer,*

$$[r(\cos t+i\sin t)]^n = r^n(\cos nt+i\sin nt).$$

Recall that this theorem is the basis for proving the following corollary from trigonometry, also listed here for convenience, see Problem 17 on Page 159.

Corollary 8.10.2 *Let z be a non zero complex number and let k be a positive integer. Then there are always exactly k k^{th} roots of z in $\mathbb{C}$.*

Lemma 8.10.3 *Let $a_k \in \mathbb{C}$ for $k=1,\cdots,n$ and let $p(z) \equiv \sum_{k=1}^n a_k z^k$. Then p is continuous.*

Proof:

$$|az^n - aw^n| \leq |a|\,|z-w|\,\left|z^{n-1}+z^{n-2}w+\cdots+w^{n-1}\right|.$$

Then for $|z-w|<1$, the triangle inequality implies $|w|<1+|z|$ and so if $|z-w|<1$,

$$|az^n - aw^n| \leq |a|\,|z-w|\,n(1+|z|)^n.$$

If $\varepsilon>0$ is given, let

$$\delta < \min\left(1, \frac{\varepsilon}{|a|\,n(1+|z|)^n}\right).$$

It follows from the above inequality that for $|z-w|<\delta$, $|az^n-aw^n|<\varepsilon$. The function of the lemma is just the sum of functions of this sort and so it follows that it is also continuous.

Theorem 8.10.4 *(Fundamental theorem of Algebra) Let $p(z)$ be a nonconstant polynomial. Then there exists $z\in\mathbb{C}$ such that $p(z)=0$.*

Proof: Suppose not. Then

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

where $a_n \neq 0,\ n>0$. Then

$$|p(z)| \geq |a_n|\,|z|^n - \sum_{k=0}^{n-1}|a_k|\,|z|^k$$

and so

$$\lim_{|z|\to\infty} |p(z)| = \infty \tag{8.35}$$

because the $|z|^n$ term dominates all the others for large $|z|$. More precisely,

$$|p(z)| \geq |z|^n \left(|a_n| - \sum_{k=0}^{n-1} |a_k| \frac{|z|^k}{|z|^n} \right)$$

and since $n > k$, all those terms in the sum are small for large $|z|$. Now let

$$\lambda \equiv \inf \{|p(z)| : z \in \mathbb{C}\}.$$

By 8.35, there exists an $R > 0$ such that if $|z| > R$, it follows that $|p(z)| > \lambda + 1$.

Therefore,

$$\begin{aligned} \lambda &\equiv \inf \{|p(z)| : z \in \mathbb{C}\} = \inf \{|p(z)| : |z| \leq R\} \\ &= \inf \left\{|p(x+iy)| : x^2 + y^2 \leq R^2\right\} \\ &\geq \inf \{|p(x+iy)| : |x| \leq R \text{ and } |y| \leq R\} \\ &\geq \inf \{|p(z)| : z \in \mathbb{C}\} \equiv \lambda \end{aligned}$$

By Theorem 6.3.3 on Page 98 there exists w such that $|p(w)| = \lambda$. A contradiction is obtained if $|p(w)| = 0$ so assume $|p(w)| > 0$. Then consider

$$q(z) \equiv \frac{p(z+w)}{p(w)}.$$

It follows $q(z)$ is of the form

$$q(z) = 1 + c_k z^k + \cdots + c_n z^n$$

where $c_k \neq 0$. The reason the constant term is 1 is because $q(0) = 1$. It is also true that $|q(z)| \geq 1$ by the assumption that $|p(w)|$ is the smallest value of $|p(z)|$. Now let $\theta \in \mathbb{C}$ be a complex number with $|\theta| = 1$ and

$$\theta c_k w^k = -|w|^k |c_k|.$$

If

$$w \neq 0, \theta = \frac{-\left|w^k\right| |c_k|}{w^k c_k}$$

and if $w = 0$, $\theta = 1$ will work. Now let $\eta^k = \theta$ so η is a k^{th} root of θ and let t be a small positive number.

$$q(t\eta w) \equiv 1 - t^k |w|^k |c_k| + \cdots + c_n t^n (\eta w)^n$$

which is of the form

$$1 - t^k |w|^k |c_k| + t^k (g(t,w))$$

where $\lim_{t \to 0} g(t,w) = 0$. Letting t be small enough this yields a contradiction to $|q(z)| \geq 1$ because eventually, for small enough t,

$$|g(t,w)| < |w|^k |c_k| / 2$$

and so for such t,

$$|q(t\eta w)| < 1 - t^k |w|^k |c_k| + t^k |w|^k |c_k| / 2 < 1,$$

This proves the theorem.

8.11 Some Other Theorems

First recall Theorem 5.4.8 on Page 84. For convenience, the version of this theorem which is of interest here is listed below.

Theorem 8.11.1 *Suppose $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ both converge absolutely. Then*

$$\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Furthermore, $\sum_{n=0}^{\infty} c_n$ converges absolutely.

Proof: It only remains to verify the last series converges absolutely. Letting p_{nk} equal 1 if $k \leq n$ and 0 if $k > n$. Then by Theorem 5.4.5 on Page 82

$$\begin{aligned}
\sum_{n=0}^{\infty} |c_n| &= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k| |b_{n-k}| = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_{nk} |a_k| |b_{n-k}| \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_{nk} |a_k| |b_{n-k}| = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} |a_k| |b_{n-k}| \\
&= \sum_{k=0}^{\infty} |a_k| \sum_{n=0}^{\infty} |b_n| < \infty.
\end{aligned}$$

This proves the theorem.

The above theorem is about multiplying two series. What if you wanted to consider

$$\left(\sum_{n=0}^{\infty} a_n\right)^p$$

where p is a positive integer maybe larger than 2? Is there a similar theorem to the above?

Definition 8.11.2 *Define*

$$\sum_{k_1+\cdots+k_p=m} a_{k_1} a_{k_2} \cdots a_{k_p}$$

as follows. Consider all ordered lists of nonnegative integers $k_1, \cdots, k_p$ which have the property that $\sum_{i=1}^{p} k_i = m$. For each such list of integers, form the product, $a_{k_1} a_{k_2} \cdots a_{k_p}$ and then add all these products.

Note that

$$\sum_{k=0}^{n} a_k a_{n-k} = \sum_{k_1+k_2=n} a_{k_1} a_{k_2}$$

Therefore, from the above theorem, if $\sum a_i$ converges absolutely, it follows

$$\left(\sum_{i=0}^{\infty} a_i\right)^2 = \sum_{n=0}^{\infty}\left(\sum_{k_1+k_2=n} a_{k_1}a_{k_2}\right).$$

It turns out a similar theorem holds for replacing 2 with p.

Theorem 8.11.3 *Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely. Then if p is a positive integer,*

$$\left(\sum_{n=0}^{\infty} a_n\right)^p = \sum_{m=0}^{\infty} c_{mp}$$

where

$$c_{mp} \equiv \sum_{k_1+\cdots+k_p=m} a_{k_1}\cdots a_{k_p}.$$

Proof: First note this is obviously true if $p = 1$ and is also true if $p = 2$ from the above theorem. Now suppose this is true for p and consider $\left(\sum_{n=0}^{\infty} a_n\right)^{p+1}$. By the induction hypothesis and the above theorem on the Cauchy product,

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n\right)^{p+1} &= \left(\sum_{n=0}^{\infty} a_n\right)^p\left(\sum_{n=0}^{\infty} a_n\right) \\
&= \left(\sum_{m=0}^{\infty} c_{mp}\right)\left(\sum_{n=0}^{\infty} a_n\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_{kp}a_{n-k}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{k_1+\cdots+k_p=k} a_{k_1}\cdots a_{k_p}a_{n-k} \\
&= \sum_{n=0}^{\infty}\sum_{k_1+\cdots+k_{p+1}=n} a_{k_1}\cdots a_{k_{p+1}}
\end{aligned}$$

and this proves the theorem.

This theorem implies the following corollary for power series.

Corollary 8.11.4 *Let*

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

be a power series having radius of convergence, $r > 0$. Then if $|x-a| < r$,

$$\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right)^p = \sum_{n=0}^{\infty} b_{np}(x-a)^n$$

where

$$b_{np} \equiv \sum_{k_1+\cdots+k_p=n} a_{k_1}\cdots a_{k_p}.$$

Proof: Since $|x-a|<r$, the series, $\sum_{n=0}^{\infty} a_n (x-a)^n$, converges absolutely. Therefore, the above theorem applies and

$$\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right)^p =$$

$$\sum_{n=0}^{\infty}\left(\sum_{k_1+\cdots+k_p=n} a_{k_1}(x-a)^{k_1}\cdots a_{k_p}(x-a)^{k_p}\right) =$$

$$\sum_{n=0}^{\infty}\left(\sum_{k_1+\cdots+k_p=n} a_{k_1}\cdots a_{k_p}\right)(x-a)^n .$$

With this theorem it is possible to consider the question raised in Example 8.8.3 on Page 167 about the existence of the power series for $\tan x$. This question is clearly included in the more general question of when

$$\left(\sum_{n=0}^{\infty} a_n (x-a)^n\right)^{-1}$$

has a power series.

Lemma 8.11.5 *Let $f(x)=\sum_{n=0}^{\infty} a_n (x-a)^n$, a power series having radius of convergence $r>0$. Suppose also that $f(a)=1$. Then there exists $r_1>0$ and $\{b_n\}$ such that for all $|x-a|<r_1$,*

$$\frac{1}{f(x)}=\sum_{n=0}^{\infty} b_n (x-a)^n .$$

Proof: By continuity, there exists $r_1>0$ such that if $|x-a|<r_1$, then

$$\sum_{n=1}^{\infty} |a_n|\,|x-a|^n<1.$$

Now pick such an x. Then

$$\begin{aligned}\frac{1}{f(x)} &= \frac{1}{1+\sum_{n=1}^{\infty} a_n (x-a)^n}\\ &= \frac{1}{1+\sum_{n=0}^{\infty} c_n (x-a)^n}\end{aligned}$$

where $c_n=a_n$ if $n>0$ and $c_0=0$. Then

$$\left|\sum_{n=1}^{\infty} a_n (x-a)^n\right| \leq \sum_{n=1}^{\infty} |a_n|\,|x-a|^n<1 \tag{8.36}$$

and so from the formula for the sum of a geometric series,

$$\frac{1}{f(x)}=\sum_{p=0}^{\infty}\left(-\sum_{n=0}^{\infty} c_n (x-a)^n\right)^p .$$

By Corollary 8.11.4, this equals

$$\sum_{p=0}^{\infty}\sum_{n=0}^{\infty} b_{np}(x-a)^n \tag{8.37}$$

where

$$b_{np} = \sum_{k_1+\cdots+k_p=n} (-1)^p c_{k_1}\cdots c_{k_p}.$$

Thus $|b_{np}| \leq \sum_{k_1+\cdots+k_p=n} |c_{k_1}|\cdots|c_{k_p}| \equiv B_{np}$ and so by Theorem 8.11.3,

$$\begin{aligned}\sum_{p=0}^{\infty}\sum_{n=0}^{\infty} |b_{np}|\,|x-a|^n &\leq \sum_{p=0}^{\infty}\sum_{n=0}^{\infty} B_{np}\,|x-a|^n \\ &= \sum_{p=0}^{\infty}\left(\sum_{n=0}^{\infty}|c_n|\,|x-a|^n\right)^p < \infty\end{aligned}$$

by 8.36 and the formula for the sum of a geometric series. Since the series of 8.37 converges absolutely, Theorem 5.4.5 on Page 82 implies the series in 8.37 equals

$$\sum_{n=0}^{\infty}\left(\sum_{p=0}^{\infty} b_{np}\right)(x-a)^n$$

and so, letting $\sum_{p=0}^{\infty} b_{np} \equiv b_n$, this proves the lemma.

With this lemma, the following theorem is easy to obtain.

Theorem 8.11.6 *Let $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, a power series having radius of convergence $r > 0$. Suppose also that $f(a) \neq 0$. Then there exists $r_1 > 0$ and $\{b_n\}$ such that for all $|x-a| < r_1$,*

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} b_n (x-a)^n.$$

Proof: Let $g(x) \equiv f(x)/f(a)$ so that $g(x)$ satisfies the conditions of the above lemma. Then by that lemma, there exists $r_1 > 0$ and a sequence, $\{b_n\}$ such that

$$\frac{f(a)}{f(x)} = \sum_{n=0}^{\infty} b_n (x-a)^n$$

for all $|x-a| < r_1$. Then

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} \widetilde{b_n}(x-a)^n$$

where $\widetilde{b_n} = b_n/f(a)$. This proves the theorem.

There is a very interesting question related to r_1 in this theorem. Consider $f(x) = 1+x^2$. In this case $r = \infty$ but the power series for $1/f(x)$ converges only if $|x| < 1$. What happens is this, $1/f(x)$ will have a power series that will converge for $|x-a| < r_1$ where r_1 is the distance between a and the nearest singularity or zero of $f(x)$ in the complex plane. In the case of $f(x) = 1+x^2$ this function has a zero at $x = \pm i$. This is just another instance of why the natural setting for the study of power series is the complex plane. To read more on power series, you should see the book by Apostol [3] or any text on complex variable.

The Riemann And Riemann Stieltjes Integrals

The integral originated in attempts to find areas of various shapes and the ideas involved in finding integrals are much older than the ideas related to finding derivatives. In fact, Archimedes[1] was finding areas of various curved shapes about 250 B.C. using the main ideas of the integral. What is presented here is a generalization of these ideas. The main interest is in the Riemann integral but if it is easy to generalize to the so called Riemann Stieltjes integral in which the length of an interval, $[x, y]$ is replaced with an expression of the form $F(y) - F(x)$ where F is an increasing function, then the generalization is given. However, there is much more that can be written about Stieltjes integrals than what is presented here. A good source for this is the book by Apostol, [2].

9.1 The Darboux Stieltjes Integral

9.1.1 Upper And Lower Darboux Stieltjes Sums

The Darboux integral pertains to bounded functions which are defined on a bounded interval. Let $[a, b]$ be a closed interval. A set of points in $[a, b]$, $\{x_0, \cdots, x_n\}$ is a partition if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Such partitions are denoted by P or Q. For f a bounded function defined on $[a, b]$, let

$$M_i(f) \equiv \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$
$$m_i(f) \equiv \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

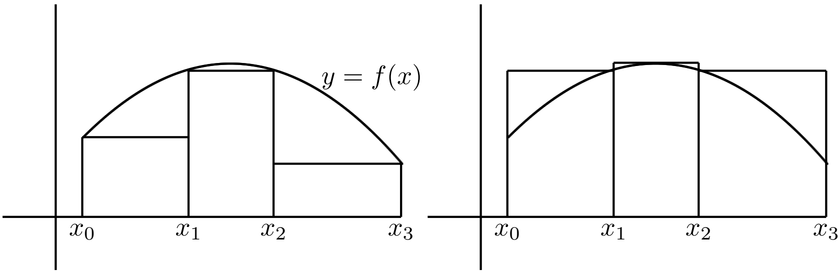
Definition 9.1.1 *Let F be an increasing function defined on $[a, b]$ and let $\Delta F_i \equiv F(x_i) - F(x_{i-1})$. Then define upper and lower sums as*

$$U(f, P) \equiv \sum_{i=1}^{n} M_i(f) \Delta F_i \text{ and } L(f, P) \equiv \sum_{i=1}^{n} m_i(f) \Delta F_i$$

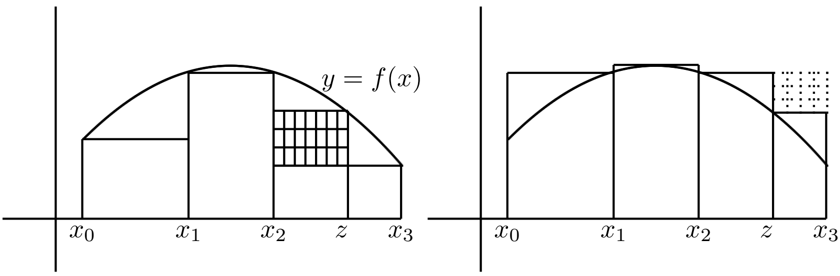
respectively. The numbers, $M_i(f)$ and $m_i(f)$, are well defined real numbers because f is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S = \{f(x) : x \in [x_{i-1}, x_i]\}$ is bounded above and below. The function, F will be called an integrator.

[1] Archimedes 287-212 B.C. found areas of curved regions by stuffing them with simple shapes which he knew the area of and taking a limit. He also made fundamental contributions to physics. The story is told about how he determined that a gold smith had cheated the king by giving him a crown which was not solid gold as had been claimed. He did this by finding the amount of water displaced by the crown and comparing with the amount of water it should have displaced if it had been solid gold.

In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition. In these pictures the function, F is given by $F(x) = x$ and these are the ordinary upper and lower sums from calculus.



What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled z has been added in.



Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 9.1.2 *If $P \subseteq Q$ then*

$$U(f,Q) \leq U(f,P), \text{ and } L(f,P) \leq L(f,Q).$$

Proof: This is verified by adding in one point at a time. Thus let

$$P = \{x_0, \cdots, x_n\}$$

and let

$$Q = \{x_0, \cdots, x_k, y, x_{k+1}, \cdots, x_n\}.$$

Thus exactly one point, y, is added between x_k and x_{k+1}. Now the term in the upper sum which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f,P)$ is

$$\sup\{f(x) : x \in [x_k, x_{k+1}]\}(F(x_{k+1}) - F(x_k)) \tag{9.1}$$

and the term which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f,Q)$ is

$$\begin{aligned} &\sup\{f(x) : x \in [x_k, y]\}(F(y) - F(x_k)) \\ &+ \sup\{f(x) : x \in [y, x_{k+1}]\}(F(x_{k+1}) - F(y)) \\ &\equiv M_1(F(y) - F(x_k)) + M_2(F(x_{k+1}) - F(y)) \end{aligned} \tag{9.2}$$

All the other terms in the two sums coincide. Now $\sup\{f(x) : x \in [x_k, x_{k+1}]\} \geq \max(M_1, M_2)$ and so the expression in 9.2 is no larger than

$$\begin{aligned} &\sup\{f(x) : x \in [x_k, x_{k+1}]\}(F(x_{k+1}) - F(y)) \\ &+\sup\{f(x) : x \in [x_k, x_{k+1}]\}(F(y) - F(x_k)) \\ =&\sup\{f(x) : x \in [x_k, x_{k+1}]\}(F(x_{k+1}) - F(x_k)), \end{aligned}$$

the term corresponding to the interval, $[x_k, x_{k+1}]$ and $U(f, P)$. This proves the first part of the lemma pertaining to upper sums because if $Q \supseteq P$, one can obtain Q from P by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part about lower sums is similar and is left as an exercise.

Lemma 9.1.3 *If P and Q are two partitions, then*

$$L(f, P) \leq U(f, Q).$$

Proof: By Lemma 9.1.2,

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

Definition 9.1.4

$$\overline{I} \equiv \inf\{U(f, Q) \text{ where } Q \text{ is a partition}\}$$
$$\underline{I} \equiv \sup\{L(f, P) \text{ where } P \text{ is a partition}\}.$$

Note that $\underline{I}$ and $\overline{I}$ are well defined real numbers.

Theorem 9.1.5 $\underline{I} \leq \overline{I}$.

Proof: From Lemma 9.1.3,

$$\underline{I} = \sup\{L(f, P) \text{ where } P \text{ is a partition}\} \leq U(f, Q)$$

because $U(f, Q)$ is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since Q is arbitrary,

$$\begin{aligned} \underline{I} &= \sup\{L(f, P) \text{ where } P \text{ is a partition}\} \\ &\leq \inf\{U(f, Q) \text{ where } Q \text{ is a partition}\} \equiv \overline{I} \end{aligned}$$

where the inequality holds because it was just shown that $\underline{I}$ is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 9.1.6 *A bounded function f is Darboux Stieltjes integrable, written as*

$$f \in R([a, b])$$

if

$$\underline{I} = \overline{I}$$

and in this case,

$$\int_a^b f(x)\, dF \equiv \underline{I} = \overline{I}.$$

When $F(x) = x$, the integral is called the Darboux integral and is written as

$$\int_a^b f(x)\, dx.$$

Thus, in words, the Darboux integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall the following Proposition which comes from the definitions.

Proposition 9.1.7 *Let S be a nonempty set and suppose $-\infty < \sup(S) < \infty$. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ *exists, then for every $\delta > 0$,*

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

This proposition implies the following theorem which is used to determine the question of Darboux Stieltjes integrability.

Theorem 9.1.8 *A bounded function f is Darboux Stieltjes integrable if and only if for all $\varepsilon > 0$, there exists a partition P such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{9.3}$$

Proof: First assume f is Darboux Stieltjes integrable. Then let P and Q be two partitions such that

$$U(f,Q) < \overline{I} + \varepsilon/2, \ L(f,P) > \underline{I} - \varepsilon/2.$$

Then since $\underline{I} = \overline{I}$,

$$U(f, Q \cup P) - L(f, P \cup Q) \leq U(f,Q) - L(f,P) < \overline{I} + \varepsilon/2 - (\underline{I} - \varepsilon/2) = \varepsilon.$$

Now suppose that for all $\varepsilon > 0$ there exists a partition such that 9.3 holds. Then for given ε and partition P corresponding to ε

$$\overline{I} - \underline{I} \leq U(f,P) - L(f,P) \leq \varepsilon.$$

Since ε is arbitrary, this shows $\underline{I} = \overline{I}$ and this proves the theorem.

The criterion of the above theorem which is equivalent to the existence of the Darboux Stieltjes integral will be referred to in what follows as the Riemann criterion.

Not all bounded functions are Darboux integrable. For example, let $F(x) = x$ and

$$f(x) \equiv \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{9.4}$$

Then if $[a,b] = [0,1]$ all upper sums for f equal 1 while all lower sums for f equal 0. Therefore the criterion of Theorem 9.1.8 is violated for $\varepsilon = 1/2$.

Here is an interesting theorem about change of variables [30]. First here is some notation: $f \in R[a,b;F]$ will mean f is Darboux Stieltjes integrable on $[a,b]$ and F is the integrator. Also let

$$M_i(f) \equiv \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

with

$$m_i(f) \equiv \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

Theorem 9.1.9 *Suppose ϕ is an increasing one to one and continuous function which maps $[a,b]$ onto $[A,B]$ and $f \in R[A,B;F]$. Then $f \circ \phi \in R[a,b;F \circ \phi]$ and*

$$\int_a^b f \circ \phi d(F \circ \phi) = \int_A^B f dF$$

Proof: By assumption, there exists a partition of $[A,B]$, $P=\{y_0,y_1,\cdots,y_n\}$ such that

$$\varepsilon > |U(P,f)-L(P,f)| \tag{9.5}$$

Now let $y_i=\phi(x_i)$ so that $\{x_0,x_1,\cdots,x_n\}$ is a partition of $[a,b]$, denoted by $\phi^{-1}(P)$. Also note

$$M_i(f)=M_i(f\circ\phi),\ m_i(f)=m_i(f\circ\phi)$$

and

$$\sum_{i=1}^{n} M_i(f)(F(y_i)-F(y_{i-1}))=\sum_{i=1}^{n} M_i(f\circ\phi)(F(\phi(x_i))-F(\phi(x_{i-1})))$$

with a similar conclusion holding for the lower sums. Then from 9.5,

$$\begin{aligned}
\varepsilon &> |U(P,f)-L(P,f)|=\sum_{i=1}^{n}(M_i(f)-m_i(f))(F(y_i)-F(y_{i-1})) \\
&= \sum_{i=1}^{n}(M_i(f\circ\phi)-m_i(f\circ\phi))(F(\phi(x_i))-F(\phi(x_{i-1}))) \\
&= \left|U\left(\phi^{-1}(P),f\circ\phi\right)-L\left(\phi^{-1}(P),f\circ\phi\right)\right|
\end{aligned}$$

which shows by the Riemann criterion that $f\circ\phi\in R[a,b;F\circ\phi]$. Also both

$$\int_a^b f\circ\phi d(F\circ\phi),\ \int_A^B fdF$$

are in the same interval of length ε,

$$\left[L\left(\phi^{-1}(P),f\circ\phi\right),U\left(\phi^{-1}(P),f\circ\phi\right)\right]$$

and so since ε is arbitrary, this shows the two integrals are the same. This proves the theorem.

9.2 Exercises

1. Prove the second half of Lemma 9.1.2 about lower sums.

2. Verify that for f given in 9.4, the lower sums on the interval $[0,1]$ are all equal to zero while the upper sums are all equal to one.

3. Let $f(x)=1+x^2$ for $x\in[-1,3]$ and let $P=\left\{-1,-\frac{1}{3},0,\frac{1}{2},1,2\right\}$. Find $U(f,P)$ and $L(f,P)$ for $F(x)=x$ and for $F(x)=x^3$.

4. Show that if $f\in R([a,b])$ for $F(x)=x$, there exists a partition, $\{x_0,\cdots,x_n\}$ such that for any $z_k\in[x_k,x_{k+1}]$,

$$\left|\int_a^b f(x)\,dx-\sum_{k=1}^{n} f(z_k)(x_k-x_{k-1})\right|<\varepsilon$$

This sum, $\sum_{k=1}^{n} f(z_k)(x_k-x_{k-1})$, is called a Riemann sum and this exercise shows that the Darboux integral can always be approximated by a Riemann sum. For the general Darboux Stieltjes case, does anything change? Explain.

5. Suppose $\{f_n\}$ is a sequence of functions which are Darboux Stieltjes integrable with respect to the integrator F on $[a,b]$. Suppose also that the sequence converges uniformly to f on $[a,b]$. Show f is also integrable and
$$\lim_{n\to\infty}\int_a^b f_n dF = \int_a^b f dF.$$

6. Let $P = \left\{1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2\right\}$ and $F(x) = x$. Find upper and lower sums for the function, $f(x) = \frac{1}{x}$ using this partition. What does this tell you about $\ln(2)$?

7. If $f \in R([a,b])$ with $F(x) = x$ and f is changed at finitely many points, show the new function is also in $R([a,b])$. Is this still true for the general case where F is only assumed to be an increasing function? Explain.

8. In the case where $F(x) = x$, define a "left sum" as
$$\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$
and a "right sum",
$$\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$
Also suppose that all partitions have the property that $x_k - x_{k-1}$ equals a constant, $(b-a)/n$ so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as n gets larger and larger. Show that for f given in 9.4, $\int_0^x f(t)\,dt = 1$ if x is rational and $\int_0^x f(t)\,dt = 0$ if x is irrational. It turns out that the correct answer should always equal zero for that function, regardless of whether x is rational. This illustrates why this method of defining the integral in terms of left and right sums is total nonsense. Show that even though this is the case, it makes no difference if f is continuous.

9. The Darboux Stieltjes integral has been defined above for F an increasing integrator function. Suppose F is an arbitrary function defined on $[a,b]$. For $P_x \equiv \{x_0, \cdots, x_n\}$ a partition of $[a,x]$,define $V(P_x, F)$ by
$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.$$
Define the total variation of F on $[a,x]$ by
$$V_{[a,x]}(F) \equiv \sup\{V(P_x,F) : P_x \text{ is a partition of } [a,x]\}.$$
Show $x \to V_{[a,x]}(F)$ is an increasing function. Then F is said to be of bounded variation on $[a,b]$ if $V_{[a,b]}(F)$ is finite. Show that every function of bounded variation can be written as the difference of two increasing functions one of which is the function $x \to V_{[a,x]}(F)$.

10. Using Problem 9, explain how to define $\int_a^b f dF$ for F a function of bounded variation.

11. The function $F(x) \equiv \lfloor x \rfloor$ gives the greatest integer less than or equal to x. Thus $F(1/2) = 0, F(5.67) = 5, F(5) = 5$, etc. If $F(x) = \lfloor x \rfloor$ as just described, find $\int_0^{10} x dF$. More generally, find $\int_0^n f(x)\,dF$ where f is a continuous function.

12. Suppose f is a bounded function on $[0,1]$ and for each $\varepsilon > 0$, f is Darboux integrable on $[\varepsilon, 1]$. Can you conclude f is Darboux integrable on $[0,1]$?

9.2.1 Functions Of Darboux Integrable Functions

It is often necessary to consider functions of Darboux integrable functions and a natural question is whether these are Darboux integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

Theorem 9.2.1 *Let f, g be bounded functions and let*

$$f([a,b]) \subseteq [c_1, d_1], \; g([a,b]) \subseteq [c_2, d_2].$$

Let $H : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R}$ satisfy,

$$|H(a_1, b_1) - H(a_2, b_2)| \leq K[|a_1 - a_2| + |b_1 - b_2|]$$

for some constant K. Then if $f, g \in R([a,b])$ it follows that $H \circ (f, g) \in R([a,b])$.

Proof: In the following claim, $M_i(h)$ and $m_i(h)$ have the meanings assigned above with respect to some partition of $[a,b]$ for the function, h.

Claim: The following inequality holds.

$$|M_i(H \circ (f,g)) - m_i(H \circ (f,g))| \leq$$

$$K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|].$$

Proof of the claim: By the above proposition, there exist $x_1, x_2 \in [x_{i-1}, x_i]$ such that

$$H(f(x_1), g(x_1)) + \eta > M_i(H \circ (f,g)),$$

and

$$H(f(x_2), g(x_2)) - \eta < m_i(H \circ (f,g)).$$

Then

$$|M_i(H \circ (f,g)) - m_i(H \circ (f,g))|$$

$$\begin{aligned} &< 2\eta + |H(f(x_1), g(x_1)) - H(f(x_2), g(x_2))| \\ &< 2\eta + K[|f(x_1) - f(x_2)| + |g(x_1) - g(x_2)|] \\ &\leq 2\eta + K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|]. \end{aligned}$$

Since $\eta > 0$ is arbitrary, this proves the claim.

Now continuing with the proof of the theorem, let P be such that

$$\sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta F_i < \frac{\varepsilon}{2K}, \; \sum_{i=1}^{n} (M_i(g) - m_i(g)) \Delta F_i < \frac{\varepsilon}{2K}.$$

Then from the claim,

$$\sum_{i=1}^{n} (M_i(H \circ (f,g)) - m_i(H \circ (f,g))) \Delta F_i$$

$$< \sum_{i=1}^{n} K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|] \Delta F_i < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows $H \circ (f,g)$ satisfies the Riemann criterion and hence $H \circ (f,g)$ is Darboux integrable as claimed. This proves the theorem.

This theorem implies that if f, g are Darboux Stieltjes integrable, then so is $af + bg, |f|, f^2$, along with infinitely many other such continuous combinations of Darboux Stieltjes integrable functions. For example, to see that $|f|$ is Darboux integrable, let $H(a,b) = |a|$. Clearly this function satisfies the conditions of the above theorem and so $|f| = H(f,f) \in R([a,b])$ as claimed. The following theorem gives an example of many functions which are Darboux Stieltjes integrable.

Theorem 9.2.2 *Let $f : [a,b] \to \mathbb{R}$ be either increasing or decreasing on $[a,b]$ and suppose F is continuous. Then $f \in R([a,b])$. If $f : [a,b] \to \mathbb{R}$ is either increasing or decreasing and continuous and no continuity requirement is made on F then $f \in R([a,b])$.*

Proof: Let $\varepsilon > 0$ be given and let

$$x_i = a + i\left(\frac{b-a}{n}\right),\ i = 0, \cdots, n.$$

Since F is continuous, it follows from Corollary 6.6.3 on Page 102, it is uniformly continuous. Therefore, if n is large enough, then for all i,

$$F(x_i) - F(x_{i-1}) < \frac{\varepsilon}{f(b) - f(a) + 1}$$

Then since f is increasing,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(F(x_i) - F(x_{i-1})) \\ &\leq \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \frac{\varepsilon}{f(b) - f(a) + 1} (f(b) - f(a)) < \varepsilon. \end{aligned}$$

Thus the Riemann criterion is satisfied and so the function is Darboux Stieltjes integrable. The proof for decreasing f is similar.

Now consider the case where f is continuous and increasing and F is only given to be increasing. Then as before, if P is a partition, $\{x_0, \cdots, x_n\}$,

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(F(x_i) - F(x_{i-1})) \tag{9.6}$$

Since f is continuous, it is uniformly continuous and so there exists $\delta > 0$ such that if $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{(F(b) - F(a)) + 1}$$

Then in 9.6, if each $|x_i - x_{i-1}| < \delta$, then

$$\begin{aligned} U(f,P) - L(f,P) &\leq \sum_{i=1}^{n} \frac{\varepsilon}{(F(b) - F(a)) + 1} (F(x_i) - F(x_{i-1})) \\ &\leq \frac{\varepsilon}{(F(b) - F(a)) + 1} (F(b) - F(a)) < \varepsilon \end{aligned}$$

Thus the Riemann criterion is satisfied and so the function is Darboux Stieltjes integrable. The proof for decreasing f is similar. This proves the theorem.

Corollary 9.2.3 *Let $[a,b]$ be a bounded closed interval and let $\phi : [a,b] \to \mathbb{R}$ be Lipschitz continuous and suppose F is increasing. Then $\phi \in R([a,b])$. Recall that a function, ϕ, is Lipschitz continuous if there is a constant, K, such that for all x, y,*

$$|\phi(x) - \phi(y)| < K|x - y|.$$

Proof: Let $f(x) = x$. Then by Theorem 9.2.2, f is Darboux Stieltjes integrable. Let $H(a,b) \equiv \phi(a)$. Then by Theorem 9.2.1 $H \circ (f, f) = \phi \circ f = \phi$ is also Darboux Stieltjes integrable. This proves the corollary.

In fact, it is enough to assume ϕ is continuous, although this is harder. This is the content of the next theorem which is where the difficult theorems about continuity and uniform continuity are used.

Theorem 9.2.4 *Suppose $f : [a,b] \to \mathbb{R}$ is continuous and F is just an increasing function defined on $[a,b]$. Then $f \in R([a,b])$.*

Proof: By Corollary 6.6.3 on Page 102, f is uniformly continuous on $[a,b]$. Therefore, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ such that if $|x_i - x_{i-1}| < \delta$, then $M_i - m_i < \frac{\varepsilon}{F(b)-F(a)+1}$. Let

$$P \equiv \{x_0, \cdots, x_n\}$$

be a partition with $|x_i - x_{i-1}| < \delta$. Then

$$\begin{aligned} U(f,P) - L(f,P) &< \sum_{i=1}^{n} (M_i - m_i)(F(x_i) - F(x_{i-1})) \\ &< \frac{\varepsilon}{F(b) - F(a) + 1}(F(b) - F(a)) < \varepsilon. \end{aligned}$$

By the Riemann criterion, $f \in R([a,b])$. This proves the theorem.

9.2.2 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.

Lemma 9.2.5 *Let S be a nonempty set which is bounded above and below. Then if $-S \equiv \{-x : x \in S\}$,*

$$\sup(-S) = -\inf(S) \tag{9.7}$$

and

$$\inf(-S) = -\sup(S). \tag{9.8}$$

Proof: Consider 9.7. Let $x \in S$. Then $-x \leq \sup(-S)$ and so $x \geq -\sup(-S)$. If follows that $-\sup(-S)$ is a lower bound for S and therefore, $-\sup(-S) \leq \inf(S)$. This implies $\sup(-S) \geq -\inf(S)$. Now let $-x \in -S$. Then $x \in S$ and so $x \geq \inf(S)$ which implies $-x \leq -\inf(S)$. Therefore, $-\inf(S)$ is an upper bound for $-S$ and so $-\inf(S) \geq \sup(-S)$. This shows 9.7. Formula 9.8 is similar and is left as an exercise.

In particular, the above lemma implies that for $M_i(f)$ and $m_i(f)$ defined above $M_i(-f) = -m_i(f)$, and $m_i(-f) = -M_i(f)$.

Lemma 9.2.6 *If $f \in R([a,b])$ then $-f \in R([a,b])$ and*

$$-\int_a^b f(x)\, dF = \int_a^b -f(x)\, dF.$$

Proof: The first part of the conclusion of this lemma follows from Theorem 9.2.2 since the function $\phi(y) \equiv -y$ is Lipschitz continuous. Now choose P such that

$$\int_a^b -f(x)\,dF - L(-f,P) < \varepsilon.$$

Then since $m_i(-f) = -M_i(f)$,

$$\varepsilon > \int_a^b -f(x)\,dF - \sum_{i=1}^n m_i(-f)\,\Delta F_i = \int_a^b -f(x)\,dF + \sum_{i=1}^n M_i(f)\,\Delta F_i$$

which implies

$$\varepsilon > \int_a^b -f(x)\,dF + \sum_{i=1}^n M_i(f)\,\Delta F_i \geq \int_a^b -f(x)\,dF + \int_a^b f(x)\,dF.$$

Thus, since ε is arbitrary,

$$\int_a^b -f(x)\,dF \leq -\int_a^b f(x)\,dF$$

whenever $f \in R([a,b])$. It follows

$$\int_a^b -f(x)\,dF \leq -\int_a^b f(x)\,dF = -\int_a^b -(-f(x))\,dF \leq \int_a^b -f(x)\,dF$$

and this proves the lemma.

Theorem 9.2.7 *The integral is linear,*

$$\int_a^b (\alpha f + \beta g)(x)\,dF = \alpha \int_a^b f(x)\,dF + \beta \int_a^b g(x)\,dF.$$

whenever $f, g \in R([a,b])$ *and* $\alpha, \beta \in \mathbb{R}$.

Proof: First note that by Theorem 9.2.1, $\alpha f + \beta g \in R([a,b])$. To begin with, consider the claim that if $f, g \in R([a,b])$ then

$$\int_a^b (f+g)(x)\,dF = \int_a^b f(x)\,dF + \int_a^b g(x)\,dF. \tag{9.9}$$

Let P_1,Q_1 be such that

$$U(f,Q_1) - L(f,Q_1) < \varepsilon/2,\ U(g,P_1) - L(g,P_1) < \varepsilon/2.$$

Then letting $P \equiv P_1 \cup Q_1$, Lemma 9.1.2 implies

$$U(f,P) - L(f,P) < \varepsilon/2, \text{ and } U(g,P) - U(g,P) < \varepsilon/2.$$

Next note that

$$m_i(f+g) \geq m_i(f) + m_i(g),\ M_i(f+g) \leq M_i(f) + M_i(g).$$

Therefore,

$$L(g+f,P) \geq L(f,P) + L(g,P),\ U(g+f,P) \leq U(f,P) + U(g,P).$$

For this partition,

$$\int_a^b (f+g)(x)\ dF \in [L(f+g,P), U(f+g,P)]$$
$$\subseteq [L(f,P)+L(g,P), U(f,P)+U(g,P)]$$

and

$$\int_a^b f(x)\ dF + \int_a^b g(x)\ dF \in [L(f,P)+L(g,P), U(f,P)+U(g,P)].$$

Therefore,

$$\left|\int_a^b (f+g)(x)\ dF - \left(\int_a^b f(x)\ dF + \int_a^b g(x)\ dF\right)\right| \leq$$
$$U(f,P)+U(g,P)-(L(f,P)+L(g,P)) < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

This proves 9.9 since ε is arbitrary.

It remains to show that

$$\alpha \int_a^b f(x)\ dF = \int_a^b \alpha f(x)\ dF.$$

Suppose first that $\alpha \geq 0$. Then

$$\int_a^b \alpha f(x)\ dF \equiv \sup\{L(\alpha f,P) : P \text{ is a partition}\} =$$

$$\alpha \sup\{L(f,P) : P \text{ is a partition}\} \equiv \alpha \int_a^b f(x)\ dF.$$

If $\alpha < 0$, then this and Lemma 9.2.6 imply

$$\int_a^b \alpha f(x)\ dF = \int_a^b (-\alpha)(-f(x))\ dF$$

$$= (-\alpha)\int_a^b (-f(x))\ dF = \alpha \int_a^b f(x)\ dF.$$

This proves the theorem.

In the next theorem, suppose F is defined on $[a,b] \cup [b,c]$.

Theorem 9.2.8 *If $f \in R([a,b])$ and $f \in R([b,c])$, then $f \in R([a,c])$ and*

$$\int_a^c f(x)\ dF = \int_a^b f(x)\ dF + \int_b^c f(x)\ dF. \tag{9.10}$$

Proof: Let P_1 be a partition of $[a,b]$ and P_2 be a partition of $[b,c]$ such that

$$U(f,P_i) - L(f,P_i) < \varepsilon/2,\ i = 1,2.$$

Let $P \equiv P_1 \cup P_2$. Then P is a partition of $[a,c]$ and

$$U(f,P) - L(f,P)$$
$$= U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) < \varepsilon/2+\varepsilon/2 = \varepsilon. \tag{9.11}$$

Thus, $f \in R([a,c])$ by the Riemann criterion and also for this partition,

$$\int_a^b f(x)\,dF + \int_b^c f(x)\,dF \in [L(f,P_1) + L(f,P_2), U(f,P_1) + U(f,P_2)]$$
$$= [L(f,P), U(f,P)]$$

and

$$\int_a^c f(x)\,dF \in [L(f,P), U(f,P)].$$

Hence by 9.11,

$$\left| \int_a^c f(x)\,dF - \left(\int_a^b f(x)\,dF + \int_b^c f(x)\,dF \right) \right| < U(f,P) - L(f,P) < \varepsilon$$

which shows that since ε is arbitrary, 9.10 holds. This proves the theorem.

Corollary 9.2.9 *Let F be continuous and let $[a,b]$ be a closed and bounded interval and suppose that*

$$a = y_1 < y_2 \cdots < y_l = b$$

and that f is a bounded function defined on $[a,b]$ which has the property that f is either increasing on $[y_j, y_{j+1}]$ or decreasing on $[y_j, y_{j+1}]$ for $j = 1, \cdots, l-1$. Then $f \in R([a,b])$.

Proof: This follows from Theorem 9.2.8 and Theorem 9.2.2.

Given this corollary, can you think of functions which do not satisfy the conditions of this corollary? They certainly exist. Think of the one which was 1 on the rationals and 0 on the irrationals. This is a sick function and that is what is needed for the Darboux integral to not exist. Unfortunately, sometimes sick functions cannot be avoided.

The symbol, $\int_a^b f(x)\,dF$ when $a > b$ has not yet been defined.

Definition 9.2.10 *Let $[a,b]$ be an interval and let $f \in R([a,b])$. Then*

$$\int_b^a f(x)\,dF \equiv -\int_a^b f(x)\,dF.$$

Note that with this definition,

$$\int_a^a f(x)\,dF = -\int_a^a f(x)\,dF$$

and so

$$\int_a^a f(x)\,dF = 0.$$

Theorem 9.2.11 *Assuming all the integrals make sense,*

$$\int_a^b f(x)\,dF + \int_b^c f(x)\,dF = \int_a^c f(x)\,dF.$$

Proof: This follows from Theorem 9.2.8 and Definition 9.2.10. For example, assume

$$c \in (a,b).$$

Then from Theorem 9.2.8,

$$\int_a^c f(x)\, dF + \int_c^b f(x)\, dF = \int_a^b f(x)\, dF$$

and so by Definition 9.2.10,

$$\begin{aligned}\int_a^c f(x)\, dF &= \int_a^b f(x)\, dF - \int_c^b f(x)\, dF \\ &= \int_a^b f(x)\, dF + \int_b^c f(x)\, dF.\end{aligned}$$

The other cases are similar.

The following properties of the integral have either been established or they follow quickly from what has been shown so far.

$$\text{If } f \in R([a,b]) \text{ then if } c \in [a,b],\ f \in R([a,c]), \tag{9.12}$$

$$\int_a^b \alpha\, dF = \alpha\,(F(b) - F(a)), \tag{9.13}$$

$$\int_a^b (\alpha f + \beta g)(x)\, dF = \alpha \int_a^b f(x)\, dF + \beta \int_a^b g(x)\, dF, \tag{9.14}$$

$$\int_a^b f(x)\, dF + \int_b^c f(x)\, dF = \int_a^c f(x)\, dF, \tag{9.15}$$

$$\int_a^b f(x)\, dF \geq 0 \text{ if } f(x) \geq 0 \text{ and } a < b, \tag{9.16}$$

$$\left| \int_a^b f(x)\, dF \right| \leq \left| \int_a^b |f(x)|\, dF \right|. \tag{9.17}$$

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

$$|f(x)| - f(x) \geq 0,\ |f(x)| + f(x) \geq 0.$$

Therefore, by 9.16 and 9.14, if $a < b$,

$$\int_a^b |f(x)|\, dF \geq \int_a^b f(x)\, dF$$

and

$$\int_a^b |f(x)|\, dF \geq -\int_a^b f(x)\, dF.$$

Therefore,

$$\int_a^b |f(x)|\, dF \geq \left| \int_a^b f(x)\, dF \right|.$$

If $b < a$ then the above inequality holds with a and b switched. This implies 9.17.

9.2.3 Fundamental Theorem Of Calculus

In this section $F(x) = x$ so things are specialized to the ordinary Darboux integral. With these properties 9.12 - 9.17, it is easy to prove the fundamental theorem of calculus[2]. Let $f \in R([a,b])$. Then by 9.12 $f \in R([a,x])$ for each $x \in [a,b]$. The first version of the fundamental theorem of calculus is a statement about the derivative of the function

$$x \rightarrow \int_a^x f(t)\ dt.$$

Theorem 9.2.12 *Let $f \in R([a,b])$ and let*

$$F(x) \equiv \int_a^x f(t)\ dt.$$

Then if f is continuous at $x \in [a,b]$,

$$F'(x) = f(x)$$

where the derivative refers to the right or left derivative at the endpoints.

Proof: Let $x \in [a,b]$ be a point of continuity of f and let h be small enough that $x + h \in [a,b]$. Then by using 9.15,

$$\begin{aligned} & F(x+h) - F(x) - f(x)h \\ = \ & F(x+h) - F(x) - \int_x^{x+h} f(t)\,dt + \left(\int_x^{x+h} f(t)\,dt - f(x)h\right) \\ = \ & \int_x^{x+h} f(t)\,dt - f(x)h \end{aligned}$$

I need to verify

$$\int_x^{x+h} f(t)\,dt - f(x)h = o(h).$$

Using 9.13,

$$f(x) = h^{-1}\int_x^{x+h} f(x)\ dt.$$

Therefore, by 9.17,

$$\left|\frac{1}{h}\left(\int_x^{x+h} f(t)\,dt - f(x)h\right)\right| = \left|h^{-1}\int_x^{x+h} (f(t) - f(x))\ dt\right|$$

$$\leq \left|h^{-1}\int_x^{x+h} |f(t) - f(x)|\ dt\right|.$$

Let $\varepsilon > 0$ and let $\delta > 0$ be small enough that if $|t - x| < \delta$, then

$$|f(t) - f(x)| < \varepsilon.$$

[2]This theorem is why Newton and Liebnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton's predecessor, Isaac Barrow had made some progress in this direction.

Therefore, if $|h| < \delta$, the above inequality and 9.13 shows that

$$\left|h^{-1}\left(F(x+h)-F(x)\right)-f(x)\right| \leq |h|^{-1}\varepsilon|h| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\int_x^{x+h} f(t)\,dt - f(x)\,h = o(h)$$

and so this proves the theorem.

Note this gives existence of a function, G which is a solution to the initial value problem,

$$G'(x) = f(x),\ G(a) = 0$$

whenever f is continuous. This is because of Theorem 9.2.4 which gives the existence of the integral of a continuous function.

The next theorem is also called the fundamental theorem of calculus.

Theorem 9.2.13 *Let $f \in R([a,b])$ and suppose $G'(x) = f(x)$ at every point of (a,b) and G is continuous on $[a,b]$. Then*

$$\int_a^b f(x)\ dx = G(b) - G(a). \tag{9.18}$$

Proof: Let $P = \{x_0, \cdots, x_n\}$ be a partition satisfying

$$U(f,P) - L(f,P) < \varepsilon.$$

Then

$$\begin{aligned} G(b) - G(a) &= G(x_n) - G(x_0) \\ &= \sum_{i=1}^n G(x_i) - G(x_{i-1}). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} G(b) - G(a) &= \sum_{i=1}^n G'(z_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(z_i)\Delta x_i \end{aligned}$$

where z_i is some point in (x_{i-1}, x_i). It follows, since the above sum lies between the upper and lower sums, that

$$G(b) - G(a) \in [L(f,P), U(f,P)],$$

and also

$$\int_a^b f(x)\ dx \in [L(f,P), U(f,P)].$$

Therefore,

$$\left|G(b) - G(a) - \int_a^b f(x)\ dx\right| < U(f,P) - L(f,P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, 9.18 holds. This proves the theorem.

The following notation is often used in this context. Suppose F is an antiderivative of f as just described with F continuous on $[a,b]$ and $F' = f$ on (a,b). Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \equiv F(x)\,|_a^b.$$

Definition 9.2.14 *Let f be a bounded function defined on a closed interval $[a,b]$ and let $P \equiv \{x_0, \cdots, x_n\}$ be a partition of the interval. Suppose $z_i \in [x_{i-1}, x_i]$ is chosen. Then the sum*

$$\sum_{i=1}^{n} f(z_i)(x_i - x_{i-1})$$

is known as a Riemann sum. Also,

$$||P|| \equiv \max\{|x_i - x_{i-1}| : i = 1, \cdots, n\}. \tag{9.19}$$

Proposition 9.2.15 *Suppose $f \in R([a,b])$. Then there exists a partition, $P \equiv \{x_0, \cdots, x_n\}$ with the property that for any choice of $z_k \in [x_{k-1}, x_k]$,*

$$\left|\int_a^b f(x)\,dx - \sum_{k=1}^{n} f(z_k)(x_k - x_{k-1})\right| < \varepsilon.$$

Proof: Choose P such that $U(f,P) - L(f,P) < \varepsilon$ and then both $\int_a^b f(x)\,dx$ and $\sum_{k=1}^n f(z_k)(x_k - x_{k-1})$ are contained in $[L(f,P), U(f,P)]$ and so the claimed inequality must hold. This proves the proposition.

You should think how the above proposition would change in the more general case of a Darboux Stieltjes integral. This proposition is significant because it gives a way of approximating the integral.

9.3 Exercises

1. Suppose $\{f_n\}$ is a sequence of Darboux Stieltjes integrable functions defined on a closed interval, $[a,b]$. Suppose $f_n \to f$ uniformly on $[a,b]$. Show

$$\lim_{n\to\infty} \int_a^b f_n dF = \int_a^b f dF.$$

   Included in showing this is the verification that f is Darboux Stieltjes integrable.

2. Let f, g be bounded functions and let

$$f([a,b]) \subseteq [c_1, d_1],\ g([a,b]) \subseteq [c_2, d_2].$$

   Let $H : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R}$ satisfy, the following condition at every point (x,y). For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$(x,y), (x_1, y_1) \in [c_1, d_1] \times [c_2, d_2]$$

   and $\max(|x - x_1|, |y - y_1|) < \delta$ then

$$|H(x,y) - H(x_1, y_1)| < \varepsilon$$

   Then if $f, g \in R([a,b])$ it follows that $H \circ (f,g) \in R([a,b])$. Is the foregoing statement true? Prove or disprove.

3. A differentiable function f defined on $(0,\infty)$ satisfies the following conditions.

$$f(xy) = f(x) + f(y), \; f'(1) = 1.$$

Find f and sketch its graph.

4. Does there exist a function which has two continuous derivatives but the third derivative fails to exist at any point? If so, give an example. If not, explain why.

5. Suppose f is a continuous function on $[a,b]$ and

$$\int_a^b f^2 dF = 0.$$

Show that then $f(x) = 0$ for all x.

6. Suppose f is a continuous function and

$$\int_a^b f(x) x^n dx = 0$$

for $n = 0, 1, 2, 3 \cdots$. Show using Problem 5 that $f(x) = 0$ for all x. **Hint:** You might use the Weierstrass approximation theorem.

7. Here is a function:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Show this function has a derivative at every point of $\mathbb{R}$. Does it make any sense to write

$$\int_0^1 f'(x)\, dx = f(1) - f(0) = f(1)?$$

Explain.

8. Let

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Is f Darboux integrable with respect to the integrator $F(x) = x$ on the interval $[0,1]$?

9. Recall that for a power series,

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

you could differentiate term by term on the interval of convergence. Show that if the radius of convergence of the above series is $r > 0$ and if $[a,b] \subseteq (c-r, c+r)$, then

$$\begin{aligned} &\int_a^b \sum_{k=0}^{\infty} a_k (x-c)^k dx \\ = \;& a_0 (b-a) + \sum_{k=1}^{\infty} \frac{a_k}{k} (b-c)^{k+1} - \sum_{k=1}^{\infty} \frac{a_k}{k} (a-c)^{k+1} \end{aligned}$$

In other words, you can integrate term by term.

10. Find $\sum_{k=1}^{\infty} \frac{2^{-k}}{k}$.

11. Let f be Darboux integrable on $[0,1]$. Show $x \to \int_0^x f(t)\,dt$ is continuous.

12. Suppose f,g are two functions which are continuous with continuous derivatives on $[a,b]$. Show using the fundamental theorem of calculus and the product rule the integration by parts formula. Also explain why all the terms make sense.
$$\int_a^b f'(t)\,g(t)\,dt = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f(t)\,g'(t)\,dt$$

13. Show
$$\frac{1}{1+x^2} = \sum_{k=0}^{n} (-1)^k x^{2k} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}.$$
Now use this to find a series which converges to $\arctan(1) = \pi/4$. Recall
$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt.$$
For which values of x will your series converge? For which values of x does the above description of arctan in terms of an integral make sense? Does this help to show the inferiority of power series?

14. Define $F(x) \equiv \int_0^x \frac{1}{1+t^2}\,dt$. Of course $F(x) = \arctan(x)$ as mentioned aboe but just consider this function in terms of the integral. Sketch the graph of F using only its definition as an integral. Show there exists a constant M such that $-M \leq F(x) \leq M$. Next explain why $\lim_{x\to\infty} F(x)$ exists and show this limit equals $-\lim_{x\to-\infty} F(x)$.

15. In Problem 14 let the limit defined there be denoted by $\pi/2$ and define $T(x) \equiv F^{-1}(x)$ for $x \in (-\pi/2, \pi/2)$. Show $T'(x) = 1 + T(x)^2$ and $T(0) = 0$. As part of this, you must explain why $T'(x)$ exists. For $x \in [0, \pi/2]$ let $C(x) \equiv 1/\sqrt{1+T(x)^2}$ with $C(\pi/2) = 0$ and on $[0,\pi/2]$, define $S(x)$ by $\sqrt{1-C(x)^2}$. Show both $S(x)$ and $C(x)$ are differentiable on $[0,\pi/2]$ and satisfy $S'(x) = C(x)$ and $C'(x) = -S(x)$. Find the appropriate way to define $S(x)$ and $C(x)$ on all of $[0, 2\pi]$ in order that these functions will be $\sin(x)$ and $\cos(x)$ and then extend to make the result periodic of period 2π on all of $\mathbb{R}$. Note this is a way to define the trig. functions which is independent of plane geometry and also does not use power series. See the book by Hardy, [18] for this approach.

16. Show
$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt.$$
Now use the binomial theorem to find a power series for $\arcsin(x)$.

17. The initial value problem from ordinary differential equations is of the form
$$y' = f(y),\ y(0) = y_0.$$
Suppose f is a continuous function of y. Show that a function, $t \to y(t)$ solves the above initial value problem if and only if
$$y(t) = y_0 + \int_0^t f(y(s))\,ds.$$

18. Let $p, q > 1$ and satisfy
$$\frac{1}{p} + \frac{1}{q} = 1.$$
Consider the function $x = t^{p-1}$. Then solving for t, you get $t = x^{1/(p-1)} = x^{q-1}$. Explain this. Now let $a, b \geq 0$. Sketch a picture to show why
$$\int_0^b x^{q-1} dx + \int_0^a t^{p-1} dt \geq ab.$$
Now do the integrals to obtain a very important inequality
$$\frac{b^q}{q} + \frac{a^p}{p} \geq ab.$$
When will equality hold in this inequality?

19. Suppose f, g are two Darboux integrable functions on $[a, b]$ with respect to an integrator F. Verify Holder's inequality.
$$\int_a^b |f| \, |g| \, dF \leq \left(\int_a^b |f|^p \, dF \right)^{1/p} \left(\int_a^b |g|^q \, dF \right)^{1/q}$$
Hint: Do the following. Let $A = \left(\int_a^b |f|^p \, dF \right)^{1/p}, B = \left(\int_a^b |g|^q \, dF \right)^{1/q}$. Then let
$$a = \frac{|f|}{A}, b = \frac{|g|}{B}$$
and use the wonderful inequality of Problem 18.

9.4 The Riemann Stieltjes Integral

The definition of integrability given above is also called Darboux integrability and the integral defined as the unique number which lies between all upper sums and all lower sums is called the Darboux integral . The definition of the Riemann integral in terms of Riemann sums is given next. I will show that these two integrals are often the same thing.

Definition 9.4.1 *A bounded function, f defined on $[a, b]$ is said to be Riemann Stieltjes integrable if there exists a number, I with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*
$$P \equiv \{x_0, x_1, \cdots, x_n\}$$
is any partition having $||P|| < \delta$, and $z_i \in [x_{i-1}, x_i]$,
$$\left| I - \sum_{i=1}^{n} f(z_i) (F(x_i) - F(x_{i-1})) \right| < \varepsilon.$$
The number $\int_a^b f(x) \, dF$ is defined as I.

It turns out Riemann Stieltjes and Darboux Stieltjes integrals are often the same. This was shown by Darboux. The Darboux integral is easier to understand and this is why it was presented first but the Riemann integral is the right way to look at it if you want to generalize. The next theorem shows it is always at least as easy for a function to be Darboux integrable as Riemann integrable.

Theorem 9.4.2 *If a bounded function defined on $[a, b]$ is Riemann Stieltjes integrable with respect to an increasing F in the sense of Definition 9.4.1 then it is Darboux Stieltjes integrable. Furthermore the two integrals coincide.*

Proof: Let f be a bounded function integrable in the sense of Definition 9.4.1. This implies there exists I and a partition

$$P = \{x_0, x_1, \cdots, x_n\}$$

such that whenever $z_i \in [x_{i-1}, x_i]$,

$$\left| I - \sum_{i=1}^{n} f(z_i)(F(x_i) - F(x_{i-1})) \right| < \varepsilon/3$$

It follows that for M_i and m_i defined as before,

$$\left| I - \sum_{i=1}^{n} M_i (F(x_i) - F(x_{i-1})) \right| \leq \varepsilon/3$$

$$\left| I - \sum_{i=1}^{n} m_i (F(x_i) - F(x_{i-1})) \right| \leq \varepsilon/3$$

Thus for this partition, P,

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} M_i (F(x_i) - F(x_{i-1})) - \sum_{i=1}^{n} m_i (F(x_i) - F(x_{i-1}))$$

$$= \sum_{i=1}^{n} M_i (F(x_i) - F(x_{i-1})) - I + \left(I - \sum_{i=1}^{n} m_i (F(x_i) - F(x_{i-1})) \right)$$

$$\leq \left| \sum_{i=1}^{n} M_i (F(x_i) - F(x_{i-1})) - I \right| + \left| I - \sum_{i=1}^{n} m_i (F(x_i) - F(x_{i-1})) \right|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon$$

Since ε is arbitrary, this and the Riemann criterion shows f is Darboux integrable.

It also follows $I = \int_a^b f dF$ where $\int_a^b f dF$ is the Darboux integral because

$$\begin{aligned}\left| I - \int_a^b f dF \right| &\leq \left| I - \sum_{i=1}^{n} M_i (F(x_i) - F(x_{i-1})) \right| \\ &\quad + \left| \sum_{i=1}^{n} M_i (F(x_i) - F(x_{i-1})) - \int_a^b f dF \right| \\ &\leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon\end{aligned}$$

and ε is arbitrary.

Is the converse true? If f is Darboux Stieltjes integrable, is it also Riemann Stieltjes integrable in the sense of Definition 9.4.1? The answer is often yes. The theorem will be proved using some lemmas.

Lemma 9.4.3 *Suppose f is a bounded function defined on $[a,b]$ and $|f(x)| < M$ for all $x \in [a,b]$. Let Q be a partition having n points, $\{x_0^*, \cdots, x_n^*\}$ and let P be any other partition. Then*

$$|U(f,P) - L(f,P)| \leq 2Mn\,||P_F|| + |U(f,Q) - L(f,Q)|$$

where $||P_F||$ is defined by

$$\max\{F(x_i) - F(x_{i-1}) : P = \{x_0, \cdots, x_m\}\}$$

Proof of the lemma: Let $P = \{x_0, \cdots, x_m\}$. Let I denote the set

$$I \equiv \{i : [x_{i-1}, x_i] \text{ contains some point of } Q\}$$

and $I^C \equiv \{0, \cdots, m\} \setminus I$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i \in I} (M_i - m_i)(F(x_i) - F(x_{i-1})) \\ &\quad + \sum_{i \in I^C} (M_i - m_i)(F(x_i) - F(x_{i-1})) \end{aligned}$$

In the second sum above, for each $i \in I^C$, $[x_{i-1}, x_i]$ must be contained in $\left[x_{k-1}^*, x_k^*\right]$ for some k. Therefore, each term in this second sum is no larger than a corresponding term in the sum which equals $U(f,Q) - L(f,Q)$. Therefore, the second sum is no larger than $U(f,Q) - L(f,Q)$. Now consider the first sum. Since $|f(x)| \leq M$,

$$(M_i - m_i)(F(x_i) - F(x_{i-1})) \leq 2M\,||P_F||$$

and so since each of these intervals $[x_{i-1}, x_i]$ for $i \in I$ contains at least one point of Q, there can be no more than n of these. Hence the first sum is dominated by

$$2Mn\,||P_F||.$$

This proves the lemma.

Lemma 9.4.4 *If $\varepsilon > 0$ is given and f is a Darboux integrable function defined on $[a,b]$, then there exists $\delta > 0$ such that whenever $||P_F|| < \delta$, then*

$$|U(f,P) - L(f,P)| < \varepsilon.$$

Proof of the lemma: Suppose Q is a partition such that $U(f,Q) - L(f,Q) < \varepsilon/2$. There exists such a partition because f is given to be Darboux integrable. Say Q has n intervals. Then if P is any partition such that $2Mn\,||P_F|| < \varepsilon/2$, it follows from the preceeding lemma that

$$\begin{aligned} |U(f,P) - L(f,P)| &\leq 2Mn\,||P_F|| + |U(f,Q) - L(f,Q)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves the lemma.

Theorem 9.4.5 *Suppose f is Darboux Stieltjes integrable and the integrator, F is increasing and continuous. Then f is Riemann Stieltjes integrable as described in Definition 9.4.1.*

Proof: By Lemma 9.4.4 there exists $\delta > 0$ such that if $||P_F|| < \delta$, then

$$|U(f,P) - L(f,P)| < \varepsilon.$$

By the continuity of F, there exists $\eta > 0$ such that if $||P|| < \eta$, then $||P_F|| < \delta$. Therefore, the above inequality holds for such P. Letting $\int_a^b f dF$ be the Darboux Stieltjes integral, it follows that every Riemann Stieltjes sum corresponding to $||P|| < \delta$ has the property that this sum is closer to $\int_a^b f dF$ than ε which shows f is Riemann Stieltjes integrable in the sense of Definition 9.4.1 and $I = \int_a^b f dF$. This proves the theorem.

Note this shows that the Riemann integral and the Darboux integral are completely equivalent whenever the integrator function is continuous. This is the case when the integrator function is $F(x) = x$ which is the usual Riemann integral of calculus.

Recall that a continuous function, f is Darboux Stieltjes integrable whenever the integrator is increasing, not necessarily continuous. Does the same theorem hold for Riemann Stieltjes integrability in the sense of Definition 9.4.1? It does.

Theorem 9.4.6 *Let f be continuous on $[a,b]$. Then f is Riemann Stieltjes integrable in the sense of Definition 9.4.1.*

Proof: Since f is continuous and $[a,b]$ is sequentially compact, it follows from Theorem 6.6.2 that f is uniformly continuous. Thus if $\varepsilon > 0$ is given, there exists $\delta > 0$ such that if $|x - y| < \delta$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2(F(b) - F(a) + 1)}.$$

Let $P = \{x_0, \cdots, x_n\}$ be a partition such that $||P|| < \delta$. Now if you add in a point z on the interior of I_j and consider the new partition,

$$x_0 < \cdots < x_{j-1} < z < x_j < \cdots < x_n$$

denoting it by P',

$$S(P,f) - S(P',f) = \sum_{i=1}^{j-1} (f(t_i) - f(t_i'))(F(x_i) - F(x_{i-1}))$$

$$+ f(t_j)(F(x_j) - F(x_{j-1})) - f\left(t_j'\right)(F(z) - F(x_{j-1}))$$

$$-f\left(t_{j+1}'\right)(F(x_j) - F(z)) + \sum_{i=j+1}^{n} \left(f(t_i) - f\left(t_{i+1}'\right)\right)(F(x_i) - F(x_{i-1}))$$

The term, $f(t_j)(F(x_j) - F(x_{j-1}))$ can be written as

$$f(t_j)(F(x_j) - F(x_{j-1})) = f(t_j)(F(x_j) - F(z)) + f(t_j)(F(z) - F(x_{j-1}))$$

and so, the middle terms can be written as

$$\begin{gathered} f(t_j)(F(x_j) - F(z)) + f(t_j)(F(z) - F(x_{j-1})) \\ -f\left(t_j'\right)(F(z) - F(x_{j-1})) - f\left(t_{j+1}'\right)(F(x_j) - F(z)) \end{gathered}$$

$$\begin{aligned} = \; & \left(f(t_j) - f\left(t_{j+1}'\right)\right)(F(x_j) - F(z)) \\ & + \left(f(t_j) - f\left(t_j'\right)\right)(F(z) - F(x_{j-1})) \end{aligned}$$

The absolute value of this is dominated by

$$< \frac{\varepsilon}{2\left(F\left(b\right)-F\left(a\right)+1\right)}\left(F\left(x_j\right)-F\left(x_{j-1}\right)\right)$$

This is because the various pairs of values at which f is evaluated are closer than δ. Similarly,

$$\begin{aligned}
&\left|\sum_{i=1}^{j-1}\left(f\left(t_i\right)-f\left(t_i'\right)\right)\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right)\right| \\
\leq & \sum_{i=1}^{j-1}\left|f\left(t_i\right)-f\left(t_i'\right)\right|\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right) \\
\leq & \sum_{i=1}^{j-1}\frac{\varepsilon}{2\left(F\left(b\right)-F\left(a\right)+1\right)}\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sum_{i=j+1}^{n}\left(f\left(t_i\right)-f\left(t_{i+1}'\right)\right)\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right)\right| \\
\leq & \sum_{i=j+1}^{n}\frac{\varepsilon}{2\left(F\left(b\right)-F\left(a\right)+1\right)}\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right).
\end{aligned}$$

Thus

$$\left|S\left(P,f\right)-S\left(P',f\right)\right| \leq \sum_{i=1}^{n}\frac{\varepsilon}{2\left(F\left(b\right)-F\left(a\right)+1\right)}\left(F\left(x_i\right)-F\left(x_{i-1}\right)\right) < \varepsilon/2.$$

Similar reasoning would apply if you added in two new points in the partition or more generally, any finite number of new points. You would just have to consider more exceptional terms. Therefore, if $||P|| < \delta$ and Q is any partition, then from what was just shown, you can pick the points on the interiors of the intervals any way you like and

$$\left|S\left(P,f\right)-S\left(P\cup Q,f\right)\right| < \varepsilon/2.$$

Therefore, if $||P||, ||Q|| < \delta$,

$$\begin{aligned}
\left|S\left(P,f\right)-S\left(Q,f\right)\right| &\leq \left|S\left(P,f\right)-S\left(P\cup Q,f\right)\right|+\left|S\left(P\cup Q,f\right)-S\left(Q,f\right)\right| \\
&< \varepsilon/2+\varepsilon/2=\varepsilon
\end{aligned}$$

Now consider a sequence $\varepsilon_n \to 0$. Then from what was just shown, there exist $\delta_n > 0$ such that for $||P||, ||Q|| < \delta_n$,

$$\left|S\left(P,f\right)-S\left(Q,f\right)\right| < \varepsilon_n$$

Let K_n be defined by

$$K_n \equiv \overline{\cup\left\{S\left(P,f\right) : ||P|| < \delta_n\right\}}.$$

In other words, take the closure of the set of numbers consisting of all Riemann sums, $S(P,f)$ such that $||P|| < \delta_n$. It follows from the definition, $K_n \supseteq K_{n+1}$ for all n and each K_n is closed with $\operatorname{diam}(K_n) \to 0$. Then by Theorem 4.9.20 there exists a unique $I \in \cap_{n=1}^{\infty} K_n$. Letting $\varepsilon > 0$, there exists n such that $\varepsilon_n < \varepsilon$. Then if $||P|| < \delta_n$, it follows $|S(P,f) - I| \leq \varepsilon_n < \varepsilon$. Thus f is Riemann Stieltjes integrable in the sense of Definition 9.4.1 and $I = \int_a^b f dF$. This proves the theorem.

9.4.1 Change Of Variables

The formulation of the integral in terms of limits of Riemann sums makes it easy to establish the following lemma and change of variables formula which is well known from beginning calculus courses although there it is presented in less generality.

Lemma 9.4.7 *Let f be Riemann Stieltjes integrable with an integrator function F which is increasing and differentiable on $[a,b]$ with continuous derivative. Then*

$$\int_a^b f dF = \int_a^b f F' dt$$

Proof: First note that the integral on the right makes sense because F' is continuous, hence Riemann integrable. Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that if $||P|| < \delta_1$, $P = \{x_0, \cdots, x_n\}$, then

$$\left| \int_a^b f dF - \sum_{k=1}^n f(x_{i-1}) (F(x_i) - F(x_{i-1})) \right| < \frac{\varepsilon}{3} \tag{9.20}$$

Since F' is assumed continuous, there exists δ_2 such that if $|t - x| < \delta_2$, then

$$|F'(t) - F'(x)| < \frac{\varepsilon}{3(b-a)M}$$

where $M \geq |f(t)|$ for all $t \in [a,b]$. It follows from the mean value theorem that if $|y - x| < \delta_2$, then

$$\left| \frac{F(y) - F(x)}{y - x} - F'(x) \right| < \frac{\varepsilon}{3M(b-a)}$$

and therefore,

$$|o(y-x)| =$$

$$|F(y) - F(x) - F'(x)(y - x)| < \frac{\varepsilon}{3(b-a)M} |y - x| \tag{9.21}$$

Let $||P|| < \min(\delta_1, \delta_2, \delta_3)$ where δ_3 is chosen small enough that for $||P|| < \delta_3$,

$$\left| \int_a^b f F' dt - \sum_{k=1}^n f(x_{i-1}) F'(x_{i-1})(x_i - x_{i-1}) \right| < \frac{\varepsilon}{3}.$$

Returning to 9.20,

$$\left| \sum_{k=1}^n f(x_{i-1}) (F(x_i) - F(x_{i-1})) - \sum_{k=1}^n f(x_{i-1}) F'(x_{i-1})(x_i - x_{i-1}) \right|$$

$$= \left| \sum_{k=1}^n f(x_{i-1}) o(x_i - x_{i-1}) \right| \leq \sum_{k=1}^n M \frac{\varepsilon}{3(b-a)M} (x_i - x_{i-1}) = \frac{\varepsilon}{3}.$$

Therefore, for $||P|| < \delta$,

$$\left| \int_a^b f dF - \int_a^b f F' dt \right| \leq \left| \int_a^b f dF - \sum_{k=1}^n f(x_{i-1}) (F(x_i) - F(x_{i-1})) \right|$$

$$+ \left| \sum_{k=1}^n f(x_{i-1}) (F(x_i) - F(x_{i-1})) - \sum_{k=1}^n f(x_{i-1}) F'(x_{i-1})(x_i - x_{i-1}) \right|$$

$$+\left|\sum_{k=1}^{n} f\left(x_{i-1}\right) F'\left(x_{i-1}\right)\left(x_i - x_{i-1}\right) - \int_a^b fF'dt\right|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since ε is arbitrary, this shows

$$\int_a^b fdF - \int_a^b fF'dt = 0$$

and this proves the theorem.

With this lemma, here is a change of variables formula.

Theorem 9.4.8 *Let f be Riemann integrable on $[A,B]$ where the integrator function is $F(t) = t$. Also let ϕ be one to one, and differentiable on $[a,b]$ with continuous derivative such that $\phi([a,b]) = [A,B]$. Then $f \circ \phi$ is Riemann integrable on $[a,b]$ and*

$$\int_a^b f(\phi(t))\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} fdx$$

Proof: First consider the case that ϕ is increasing. By Lemma 6.4.3 either ϕ is increasing or decreasing so there are only two cases to consider for ϕ. The case where ϕ is decreasing will be considered later. By Theorem 9.1.9,

$$\int_a^b f \circ \phi d(F \circ \phi) = \int_A^B fdF = \int_{\phi(a)}^{\phi(b)} fdF$$

where $F(t) = t$. Then from Lemma 9.4.7, this reduces to

$$\int_a^b f(\phi(t))\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} fdx.$$

This proves the theorem in the case where ϕ is increasing.

Next consider the case where ϕ is decreasing so $\phi(a) = B, \phi(b) = A$. Then $-\phi$ is increasing and maps $[a,b]$ to $[-B,-A]$. Let h be defined on $[-B,-A]$ by $h(x) \equiv f(-x)$. It follows h is Riemann integrable on $[-B,-A]$. This follows from observing Riemann sums. Furthermore it is seen in this way that

$$\int_{-B}^{-A} h(y)\,dy = \int_A^B f(x)\,dx.$$

Then applying what was just shown,

$$\begin{aligned} & -\int_a^b f(\phi(t))\,\phi'(t)\,dt \\ = & \int_a^b h(-\phi(t))\left(-\phi'(t)\right)dt \\ = & \int_{-B}^{-A} h(y)\,dy = \int_A^B f(x)\,dx \\ = & \int_{\phi(b)}^{\phi(a)} f(x)\,dx \end{aligned}$$

and so

$$\int_a^b f(\phi(t))\,\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx.$$

This proves the change of variables formula.

9.4.2 General Riemann Stieltjes Integrals

Up till now the integrator function has always been increasing. However this can easily be generalized. To do so, here is a definition of a more general kind of function which will serve as an integrator. First of all, here is some notation.

Notation 9.4.9 *Let F be an increasing integrator function. Then when f is Riemann Stieltjes integrable with respect to F on $[a,b]$, this will be written as*

$$f \in R([a,b], F)$$

with similar notation applying to more general situations about to be presented.

Definition 9.4.10 *Suppose F is an arbitrary function defined on $[a,b]$. For $P_x \equiv \{x_0, \cdots, x_n\}$ a partition of $[a,x]$,define $V(P_x, F)$ by*

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| .$$

Define the total variation of F on $[a,x]$ by

$$V_{[a,x]}(F) \equiv \sup \{V(P_x, F) : P_x \text{ is a partition of } [a,x]\} .$$

Then F is said to be of bounded variation on $[a,b]$ if $V_{[a,b]}(F)$ is finite.

Then with this definition, one has an important proposition.

Proposition 9.4.11 *Every function F of bounded variation can be written as the difference of two increasing function, one of which is the function*

$$x \to V_{[a,x]}(F)$$

Furthermore, the functions of bounded variation are exactly those functions which are the difference of two increasing functions.

Proof: Let F be of bounded variation. It is obvious from the definition that $x \to V_{[a,x]}(F)$ is an increasing function. Also

$$F(x) = V_{[a,x]}(F) - \left(V_{[a,x]}(F) - F(x)\right)$$

The first part of the proposition is proved if I can show $x \to V_{[a,x]}(F) - F(x)$ is increasing. Let $x \leq y$. Is it true that

$$V_{[a,x]}(F) - F(x) \leq V_{[a,y]}(F) - F(y)?$$

This is true if and only if

$$F(y) - F(x) \leq V_{[a,y]}(F) - V_{[a,x]}(F) \tag{9.22}$$

To show this is so, first note that

$$V(P_x, F) \leq V(Q_x, F)$$

whenever the partition $Q_x \supseteq P_x$. You demonstrate this by adding in one point at a time and using the triangle inequality. Now let P_y and P_x be partitions of $[a,y]$ and $[a,x]$ respectively such that

$$V(P_x, F) + \varepsilon > V_{[a,x]}(F),\ V(P_y, F) + \varepsilon > V_{[a,y]}(F)$$

Without loss of generality P_y contains x because from what was just shown you could add in the point x and the approximation of $V(P_y, F)$ to $V_{[a,y]}(F)$ would only be better. Then from the definition,

$$\begin{aligned} V_{[a,y]}(F) - V_{[a,x]}(F) &\geq V(P_y, F) - (V(P_x, F) + \varepsilon) \\ &\geq |F(y) - F(x)| - \varepsilon \\ &\geq F(y) - F(x) - \varepsilon \end{aligned}$$

and since ε is arbitrary, this establishes 9.22. This proves the first part of the proposition.

Now suppose

$$F(x) = F_1(x) - F_2(x)$$

where each F_i is an increasing function. Why is F of bounded variation? Letting $x < y$

$$\begin{aligned} |F(y) - F(x)| &= |F_1(y) - F_2(y) - (F_1(x) - F_2(x))| \\ &\leq (F_1(y) - F_1(x)) + (F_2(y) - F_2(x)) \end{aligned}$$

Therefore, if $P = \{x_0, \cdots, x_n\}$ is any partition of $[a, b]$

$$\begin{aligned} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &\leq \sum_{i=1}^{n} (F_1(x_i) - F_1(x_{i-1})) + (F_2(x_i) - F_2(x_{i-1})) \\ &= (F_1(b) - F_1(a)) + (F_2(b) - F_2(a)) \end{aligned}$$

and this shows $V_{[a,b]}(F) \leq (F_1(b) - F_1(a)) + (F_2(b) - F_2(a))$ so F is of bounded variation. This proves the proposition.

With this proposition, it is possible to make the following definition of the Riemann Stieltjes integral.

Definition 9.4.12 *Let F be of bounded variation on $[a, b]$ and let $F = F_1 - F_2$ where F_i is increasing. Then if $g \in R([a, b], F_i)$ for $i = 1, 2$*

$$\int_a^b g dF \equiv \int_a^b g dF_1 - \int_a^b g dF_2$$

When this happens it is convenient to write $g \in R([a, b], F)$.

Of course there is the immediate question whether the above definition is well defined.

Proposition 9.4.13 *Definition 9.4.12 is well defined. Also for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $P = \{x_0, \cdots, x_n\}$ is a partition of $[a, b]$ such that $||P|| < \delta$, then if $z_i \in [x_{i-1}, x_i]$ for $i = 1, 2 \cdots, n$, then*

$$\left| \int_a^b g dF - \sum_{i=1}^{n} g(z_i)(F(x_i) - F(x_{i-1})) \right| < \varepsilon.$$

Proof: By assumption $g \in R([a, b], F_i)$ for $i = 1, 2$. Therefore, there exists δ_i such that

$$\left| \int_a^b g dF_i - \sum_{i=1}^{n} g(z_i)(F(x_i) - F(x_{i-1})) \right| < \varepsilon/2$$

whenever $P = \{x_0, \cdots, x_n\}$ with $||P|| < \delta_i$ and $z_i \in [x_{i-1}, x_i]$. Let $0 < \delta < \min(\delta_1, \delta_2)$. Then pick a partition $P = \{x_0, \cdots, x_n\}$ such that $||P|| < \delta$. It follows

$$= \left| \sum_{i=1}^{n} g(z_i)(F(x_i) - F(x_{i-1})) - \left(\int_a^b g dF_1 - \int_a^b g dF_2 \right) \right|$$

$$= \left|\left(\sum_{i=1}^{n} g(z_i)(F_1(x_i) - F_1(x_{i-1})) - \sum_{i=1}^{n} g(z_i)(F_2(x_i) - F_2(x_{i-1}))\right) - \left(\int_a^b g dF_1 - \int_a^b g dF_2\right)\right|$$

$$\leq \left|\sum_{i=1}^{n} g(z_i)(F_1(x_i) - F_1(x_{i-1})) - \int_a^b g dF_1\right|$$

$$+\left|\sum_{i=1}^{n} g(z_i)(F_2(x_i) - F_2(x_{i-1})) - \int_a^b g dF_2\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

If $F = F_1' - F_2'$ the same argument would show that for $||P||$ small enough,

$$\left|\sum_{i=1}^{n} g(z_i)(F(x_i) - F(x_{i-1})) - \left(\int_a^b g dF_1' - \int_a^b g dF_2'\right)\right| < \varepsilon \tag{9.23}$$

Therefore, picking a partition P with $||P||$ small enough to satisfy 9.23 for F_i and F_i', it follows

$$\left|\left(\int_a^b g dF_1' - \int_a^b g dF_2'\right) - \left(\int_a^b g dF_1 - \int_a^b g dF_2\right)\right|$$

$$\leq \left|\left(\int_a^b g dF_1' - \int_a^b g dF_2'\right) - \sum_{i=1}^{n} g(z_i)(F(x_i) - F(x_{i-1}))\right|$$

$$+\left|\left(\int_a^b g dF_1 - \int_a^b g dF_2\right) - \sum_{i=1}^{n} g(z_i)(F(x_i) - F(x_{i-1}))\right| < 2\varepsilon$$

since ε is arbitrary this shows the definition is well defined and the approximation claim holds. This proves the proposition.

More generally, let f and g be two arbitrary functions defined on $[a, b]$. Then the following definition tells what it means for $f \in R([a, b], g)$. Note the above takes care of the case where integrator function is of bounded variation.

Definition 9.4.14 *$f \in R([a, b], g)$ means that there exists a number I such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$P = \{x_0, \cdots, x_n\}$$

is a partition of $[a, b]$ with $||P|| < \delta$, then whenever $z_i \in [x_{i-1}, x_i]$ for each i,

$$\left|\sum_{i=1}^{n} f(z_i)(g(x_i) - g(x_{i-1})) - I\right| < \varepsilon.$$

Then

$$I = \int_a^b f dg.$$

Now here is a general integration by parts formula. This is a very remarkable formula.

Theorem 9.4.15 *Let f, g be two functions defined on $[a, b]$. Suppose $f \in R([a, b], g)$. Then $g \in R([a, b], f)$ and the following integration by parts formula holds.*

$$\int_a^b f dg + \int_a^b g df = fg(b) - fg(a).$$

Proof: By definition there exists $\delta > 0$ such that if $||P|| < \delta$ then whenever $z_i \in [x_{i-1}, x_i]$,

$$\left| \sum_{i=1}^n f(z_i)(g(x_i) - g(x_{i-1})) - \int_a^b f dg \right| < \varepsilon$$

Pick such a partition. Now consider a sum of the form

$$\sum_{i=1}^n g(t_i)(f(x_i) - f(x_{i-1}))$$

Also notice

$$fg(b) - fg(a) = \sum_{i=1}^n fg(x_i) - fg(x_{i-1}).$$

Therefore,

$$fg(b) - fg(a) - \sum_{i=1}^n g(t_i)(f(x_i) - f(x_{i-1}))$$

$$= \sum_{i=1}^n fg(x_i) - fg(x_{i-1}) - \sum_{i=1}^n g(t_i)(f(x_i) - f(x_{i-1}))$$

$$= \sum_{i=1}^n f(x_i) g(x_i) - f(x_{i-1}) g(x_{i-1}) - g(t_i) f(x_i) + g(t_i) f(x_{i-1})$$

$$= \sum_{i=1}^n f(x_i)(g(x_i) - g(t_i)) + f(x_{i-1})(g(t_i) - g(x_{i-1}))$$

But this is just a Riemann Stieltjes sum for

$$\int_a^b f dg$$

corresponding to the partition which consists of all the x_i along with all the t_i and if P' is this partition, $||P'|| < \delta$ because it possibly more points in it than P. Therefore,

$$\left| \overbrace{fg(b) - fg(a) - \sum_{i=1}^n g(t_i)(f(x_i) - f(x_{i-1}))}^{\text{Riemann sum for } \int_a^b f dg} - \int_a^b f dg \right| < \varepsilon$$

and this has shown that from the definition, $g \in R([a, b], f)$ and

$$\int_a^b g df = fg(b) - fg(a) - \int_a^b f dg.$$

This proves the theorem.

It is an easy theorem to remember. Think something sloppy like this.

$$d(fg) = fdg + gdf$$

and so

$$fg(b) - fg(a) = \int_a^b d(fg) = \int_a^b fdg + \int_a^b gdf$$

and all you need is for at least one of these integrals on the right to make sense. Then the other automatically does and the formula follows.

When is $f \in R([a,b], g)$? It was shown above in Theorem 9.4.6 along with Definition 9.4.14 and 9.4.12 and Proposition 9.4.13 that if g is of bounded variation and f is continuous, then f is Riemann Stieltjes integrable. From the above theorem on integration by parts, this yields the following existence theorem.

Theorem 9.4.16 *Suppose f is continuous and g is of bounded variation on $[a,b]$. Then $f \in R([a,b], g)$. Also if g is of bounded variation and f is continuous, then $g \in R([a,b], f)$.*

9.5 Exercises

1. Let $F(x) = \int_{x^2}^{x^3} \frac{t^5+7}{t^7+87t^6+1}\, dt$. Find $F'(x)$.

2. Let $F(x) = \int_2^x \frac{1}{1+t^4}\, dt$. Sketch a graph of F and explain why it looks the way it does.

3. Let a and b be positive numbers and consider the function,
$$F(x) = \int_0^{ax} \frac{1}{a^2+t^2}\, dt + \int_b^{a/x} \frac{1}{a^2+t^2}\, dt.$$
Show that F is a constant.

4. Solve the following initial value problem from ordinary differential equations which is to find a function y such that
$$y'(x) = \frac{x^7+1}{x^6+97x^5+7},\ y(10) = 5.$$

5. If $F, G \in \int f(x)\, dx$ for all $x \in \mathbb{R}$, show $F(x) = G(x) + C$ for some constant, C. Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion $\int_a^b f(t)\, dt = G(b) - G(a)$ where $G'(x) = f(x)$.

6. Suppose f is continuous on $[a,b]$. Show there exists $c \in (a,b)$ such that
$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$
Hint: You might consider the function $F(x) \equiv \int_a^x f(t)\, dt$ and use the mean value theorem for derivatives and the fundamental theorem of calculus.

7. Suppose f and g are continuous functions on $[a,b]$ and that $g(x) \neq 0$ on (a,b). Show there exists $c \in (a,b)$ such that
$$f(c) \int_a^b g(x)\, dx = \int_a^b f(x) g(x)\, dx.$$
Hint: Define $F(x) \equiv \int_a^x f(t) g(t)\, dt$ and let $G(x) \equiv \int_a^x g(t)\, dt$. Then use the Cauchy mean value theorem on these two functions.

8. Consider the function
$$f(x) \equiv \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$
Is f Riemann integrable? Explain why or why not.

9. Prove the second part of Theorem 9.2.2 about decreasing functions.

10. The Riemann integral is only defined for bounded functions on bounded intervals. When f is Riemann integrable on $[a, R]$ for each $R > a$ define an "improper" integral as follows.
$$\int_a^\infty f(t)\, dt \equiv \lim_{R\to\infty} \int_a^R f(t)\, dt$$
whenever this limit exists. Show
$$\int_0^\infty \frac{\sin x}{x} dx$$
exists. Here the integrand is defined to equal 1 when $x = 0$, not that this matters.

11. Show
$$\int_0^\infty \sin\left(t^2\right) dt$$
exists.

12. The Gamma function is defined for $x > 0$ by
$$\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1} dt$$
Give a meaning to the above improper integral and show it exists. Also show
$$\Gamma(x+1) = \Gamma(x)\, x,\ \Gamma(1) = 1,$$
and for n a positive integer, $\Gamma(n+1) = n!$. **Hint:** The hard part is showing the integral exists. To do this, first show that if $f(x)$ is an increasing function which is bounded above, then $\lim_{x\to\infty} f(x)$ must exist and equal $\sup\{f(x) : x \geq 0\}$. Then show
$$\int_0^R e^{-t} t^{x-1} dt \equiv \lim_{\varepsilon\to 0+} \int_\varepsilon^R e^{-t} t^{x-1} dt$$
is an increasing function of R which is bounded above.

13. The most important of all differential equations is the first order linear equation, $y' + p(t)\, y = f(t)$ where p, f are continuous. Show the solution to the initial value problem consisting of this equation and the initial condition, $y(a) = y_a$ is
$$y(t) = e^{-P(t)} y_a + e^{-P(t)} \int_a^t e^{P(s)} f(s)\, ds,$$
where $P(t) = \int_a^t p(s)\, ds$. Give conditions under which everything is correct. **Hint:** You use the integrating factor approach. Multiply both sides by $e^{P(t)}$, verify the left side equals
$$\frac{d}{dt}\left(e^{P(t)} y(t)\right),$$
and then take the integral, $\int_a^t$ of both sides.

14. Suppose f is a continuous function which is not equal to zero on $[0, b]$. Show that

$$\int_0^b \frac{f(x)}{f(x) + f(b-x)} dx = \frac{b}{2}.$$

Hint: First change the variables to obtain the integral equals

$$\int_{-b/2}^{b/2} \frac{f(y + b/2)}{f(y + b/2) + f(b/2 - y)} dy$$

Next show by another change of variables that this integral equals

$$\int_{-b/2}^{b/2} \frac{f(b/2 - y)}{f(y + b/2) + f(b/2 - y)} dy.$$

Thus the sum of these equals b.

15. Let there be three equally spaced points, $x_{i-1}, x_{i-1} + h \equiv x_i$, and $x_i + 2h \equiv x_{i+1}$. Suppose also a function, f, has the value f_{i-1} at x, f_i at $x + h$, and f_{i+1} at $x + 2h$. Then consider

$$g_i(x) \equiv \frac{f_{i-1}}{2h^2}(x - x_i)(x - x_{i+1}) - \frac{f_i}{h^2}(x - x_{i-1})(x - x_{i+1}) + \frac{f_{i+1}}{2h^2}(x - x_{i-1})(x - x_i).$$

Check that this is a second degree polynomial which equals the values f_{i-1}, f_i, and f_{i+1} at the points x_{i-1}, x_i, and x_{i+1} respectively. The function, g_i is an approximation to the function, f on the interval $[x_{i-1}, x_{i+1}]$. Also,

$$\int_{x_{i-1}}^{x_{i+1}} g_i(x)\, dx$$

is an approximation to $\int_{x_{i-1}}^{x_{i+1}} f(x)\, dx$. Show $\int_{x_{i-1}}^{x_{i+1}} g_i(x)\, dx$ equals

$$\frac{hf_{i-1}}{3} + \frac{hf_i 4}{3} + \frac{hf_{i+1}}{3}.$$

Now suppose n is even and $\{x_0, x_1, \cdots, x_n\}$ is a partition of the interval, $[a, b]$ and the values of a function, f defined on this interval are $f_i = f(x_i)$. Adding these approximations for the integral of f on the succession of intervals,

$$[x_0, x_2], [x_2, x_4], \cdots, [x_{n-2}, x_n],$$

show that an approximation to $\int_a^b f(x)\, dx$ is

$$\frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_2 + \cdots + 4f_{n-1} + f_n].$$

This is called Simpson's rule. Use Simpson's rule to compute an approximation to $\int_1^2 \frac{1}{t} dt$ letting $n = 4$.

16. Suppose $x_0 \in (a, b)$ and that f is a function which has $n+1$ continuous derivatives on this interval. Consider the following.

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^x f'(t)\, dt \\ &= f(x_0) + (t - x) f'(t)|_{x_0}^x + \int_{x_0}^x (x - t) f''(t)\, dt \\ &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x - t) f''(t)\, dt. \end{aligned}$$

Explain the above steps and continue the process to eventually obtain Taylor's formula,

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\, dt$$

where $n! \equiv n(n-1)\cdots 3\cdot 2\cdot 1$ if $n \geq 1$ and $0! \equiv 1$.

17. In the above Taylor's formula, use Problem 7 on Page 206 to obtain the existence of some z between x_0 and x such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1}.$$

Hint: You might consider two cases, the case when $x > x_0$ and the case when $x < x_0$.

18. There is a general procedure for constructing methods of approximate integration like the trapezoid rule and Simpson's rule. Consider $[0,1]$ and divide this interval into n pieces using a uniform partition, $\{x_0, \cdots, x_n\}$ where $x_i - x_{i-1} = 1/n$ for each i. The approximate integration scheme for a function, f, will be of the form

$$\left(\frac{1}{n}\right)\sum_{i=0}^{n} c_i f_i \approx \int_0^1 f(x)\, dx$$

where $f_i = f(x_i)$ and the constants, c_i are chosen in such a way that the above sum gives the exact answer for $\int_0^1 f(x)\, dx$ where $f(x) = 1, x, x^2, \cdots, x^n$. When this has been done, change variables to write

$$\begin{aligned}\int_a^b f(y)\, dy &= (b-a)\int_0^1 f(a + (b-a)x)\, dx \\ &\approx \frac{b-a}{n}\sum_{i=1}^{n} c_i f\left(a + (b-a)\left(\frac{i}{n}\right)\right) \\ &= \frac{b-a}{n}\sum_{i=1}^{n} c_i f_i\end{aligned}$$

where $f_i = f\left(a + (b-a)\left(\frac{i}{n}\right)\right)$. Consider the case where $n = 1$. It is necessary to find constants c_0 and c_1 such that

$$\begin{aligned} c_0 + c_1 = 1 = \textstyle\int_0^1 1\, dx \\ 0c_0 + c_1 = 1/2 = \textstyle\int_0^1 x\, dx. \end{aligned}$$

Show that $c_0 = c_1 = 1/2$, and that this yields the trapezoid rule. Next take $n = 2$ and show the above procedure yields Simpson's rule. Show also that if this integration scheme is applied to any polynomial of degree 3 the result will be exact. That is,

$$\frac{1}{2}\left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right) = \int_0^1 f(x)\, dx$$

whenever $f(x)$ is a polynomial of degree three. Show that if f_i are the values of f at $a, \frac{a+b}{2}$, and b with $f_1 = f\left(\frac{a+b}{2}\right)$, it follows that the above formula gives $\int_a^b f(x)\, dx$ exactly whenever f is a polynomial of degree three. Obtain an integration scheme for $n = 3$.

19. Let f have four continuous derivatives on $[x_{i-1}, x_{i+1}]$ where $x_{i+1} = x_{i-1} + 2h$ and $x_i = x_{i-1} + h$. Show using Problem 17, there exists a polynomial of degree three, $p_3(x)$, such that

$$f(x) = p_3(x) + \frac{1}{4!} f^{(4)}(\xi)(x - x_i)^4$$

Now use Problem 18 and Problem 15 to conclude

$$\left| \int_{x_{i-1}}^{x_{i+1}} f(x)\, dx - \left(\frac{hf_{i-1}}{3} + \frac{hf_i 4}{3} + \frac{hf_{i+1}}{3} \right) \right| < \frac{M}{4!} \frac{2h^5}{5},$$

where M satisfies, $M \geq \max\left\{ \left| f^{(4)}(t) \right| : t \in [x_{i-1}, x_i] \right\}$. Now let $S(a, b, f, 2m)$ denote the approximation to $\int_a^b f(x)\, dx$ obtained from Simpson's rule using $2m$ equally spaced points. Show

$$\left| \int_a^b f(x)\, dx - S(a, b, f, 2m) \right| < \frac{M}{1920} (b - a)^5 \frac{1}{m^4}$$

where $M \geq \max\left\{ \left| f^{(4)}(t) \right| : t \in [a, b] \right\}$. Better estimates are available in numerical analysis books. However, these also have the error in the form $C\left(1/m^4\right)$.

20. A **regular Sturm Liouville problem** involves the differential equation, for an unknown function of x which is denoted here by y,

$$(p(x) y')' + (\lambda q(x) + r(x)) y = 0, \; x \in [a, b]$$

and it is assumed that $p(t), q(t) > 0$ for any t along with boundary conditions,

$$\begin{aligned} C_1 y(a) + C_2 y'(a) &= 0 \\ C_3 y(b) + C_4 y'(b) &= 0 \end{aligned}$$

where

$$C_1^2 + C_2^2 > 0, \text{ and } C_3^2 + C_4^2 > 0.$$

There is an imense theory connected to these important problems. The constant, λ is called an eigenvalue. Show that if y is a solution to the above problem corresponding to$\lambda = \lambda_1$ and if z is a solution corresponding to $\lambda = \lambda_2 \neq \lambda_1$, then

$$\int_a^b q(x) y(x) z(x)\, dx = 0. \tag{9.24}$$

Hint: Do something like this:

$$\begin{aligned} (p(x) y')' z + (\lambda_1 q(x) + r(x)) yz &= 0, \\ (p(x) z')' y + (\lambda_2 q(x) + r(x)) zy &= 0. \end{aligned}$$

Now subtract and either use integration by parts or show

$$(p(x) y')' z - (p(x) z')' y = ((p(x) y') z - (p(x) z') y)'$$

and then integrate. Use the boundary conditions to show that $y'(a) z(a) - z'(a) y(a) = 0$ and $y'(b) z(b) - z'(b) y(b) = 0$. The formula, 9.24 is called an orthogonality relation and it makes possible an expansion in terms of certain functions called eigenfunctions.

21. Letting $[a,b] = [-\pi,\pi]$, consider an example of a regular Sturm Liouville problem which is of the form
$$y'' + \lambda y = 0, y(-\pi) = 0, y(\pi) = 0.$$
Show that if $\lambda = n^2$ and $y_n(x) = \sin(nx)$ for n a positive integer, then y_n is a solution to this regular Sturm Liouville problem. In this case, $q(x) = 1$ and so from Problem 20, it must be the case that
$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0$$
if $n \neq m$. Show directly using integration by parts that the above equation is true.

22. Let f be increasing and g continuous on $[a,b]$. Then there exists $c \in [a,b]$ such that
$$\int_a^b g df = g(c)(f(b) - f(c)).$$
Hint: First note g Riemann Stieltjes integrable because it is continuous. Since g is continuous, you can let
$$m = \min\{g(x) : x \in [a,b]\}$$
and
$$M = \max\{g(x) : x \in [a,b]\}$$
Then
$$m\int_a^b df \leq \int_a^b g df \leq M\int_a^b df$$
Now if $f(b) - f(a) \neq 0$, you could divide by it and conclude
$$m \leq \frac{\int_a^b g df}{f(b) - f(a)} \leq M.$$
You need to explain why $\int_a^b df = f(b) - f(a)$. Next use the intermediate value theorem to get the term in the middle equal to $g(c)$ for some c. What happens if $f(b) - f(a) = 0$? Modify the argument and fill in the details to show the conclusion still follows.

23. Suppose g is increasing and f is continuous and of bounded variation. By Theorem 9.4.16,
$$g \in R([a,b], f).$$
Show there exists $c \in [a,b]$ such that
$$\int_a^b g df = g(a)\int_a^c df + g(b)\int_c^b df$$
This is called the second mean value theorem for integrals. **Hint:** Use integration by parts.
$$\int_a^b g df = -\int_a^b f dg + f(b)g(b) - f(a)g(a)$$
Now use the first mean value theorem, the result of Problem 22 to substitute something for $\int_a^b f dg$ and then simplify.

24. Let $f : [a,b] \times [c,d] \to \mathbb{R}$ satisfy the following condition at $(x_0, y_0) \in [a,b] \times [c,d]$. For every $\varepsilon > 0$ there exists a $\delta > 0$ possibly depending on (x_0, y_0) such that if
$$\max\left(\left|x - x_0\right|, \left|y - y_0\right|\right) < \delta$$
then
$$\left|f(x,y) - f(x_0, y_0)\right| < \varepsilon.$$
This is what it means for f to be continuous at (x_0, y_0). Show that if f is continuous at every point of $[a,b] \times [c,d]$, then it is uniformly continuous on $[a,b] \times [c,d]$. That is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $(x_0, y_0), (x, y)$ are any two points of $[a,b] \times [c,d]$ such that
$$\max\left(\left|x - x_0\right|, \left|y - y_0\right|\right) < \delta,$$
then
$$\left|f(x,y) - f(x_0, y_0)\right| < \varepsilon.$$
Also show that such a function achieves its maximum and its minimum on $[a,b] \times [c,d]$. **Hint:** This is easy if you follow the same procedure that was used earlier but you take subsequences for each component to show $[a,b] \times [c,d]$ is sequentially compact.

25. Suppose f is a real valued function defined on $[a,b] \times [c,d]$ which is uniformly continuous as described in Problem 24 and bounded which follow from an assumption that it is continuous. Also suppose α, β are increasing functions. Show
$$x \to \int_c^d f(x,y)\, d\beta(y), \; y \to \int_a^b f(x,y)\, d\alpha(x)$$
are both continuous functions. The idea is you fix one of the variables, x in the first and then integrate the continuous function of y obtaining a real number which depends on the value of x fixed. Explain why it makes sense to write
$$\int_a^b \int_c^d f(x,y)\, d\beta(y)\, d\alpha(x), \; \int_c^d \int_a^b f(x,y)\, d\alpha(x)\, d\beta(y).$$
Now consider the first of the above iterated integrals. (That is what these are called.) Consider the following argument in which you fill in the details.
$$\int_a^b \int_c^d f(x,y)\, d\beta(y)\, d\alpha(x) = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \int_c^d f(x,y)\, d\beta(y)\, d\alpha(x)$$
$$= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \sum_{j=1}^m \int_{y_{j-1}}^{y_j} f(x,y)\, d\beta(y)\, d\alpha(x)$$
$$= \sum_{i=1}^n \sum_{j=1}^m \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)\, d\beta(y)\, d\alpha(x)$$
$$= \sum_{i=1}^n \sum_{j=1}^m \int_{x_{i-1}}^{x_i} \left(\beta(y_j) - \beta(y_{j-1})\right) f(x, t_j)\, d\alpha(x)$$
$$= \sum_{i=1}^n \sum_{j=1}^m \left(\beta(y_j) - \beta(y_{j-1})\right)\left(\alpha(x_i) - \alpha(x_{j-1})\right) f(s_j, t_j)$$

Also

$$\int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y)$$
$$= \sum_{j=1}^{m}\sum_{i=1}^{n} (\beta(y_j) - \beta(y_{j-1}))(\alpha(x_i) - \alpha(x_{j-1})) f\left(s_j', t_j'\right)$$

and now because of ?? it follows

$$\left|f\left(s_j', t_j'\right) - f(s_j, t_j)\right| < \frac{\varepsilon}{(\beta(d) - \beta(c))(\alpha(b) - \alpha(a))}$$

and so

$$\left|\int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y) - \int_c^d \int_a^b f(x,y)\,d\alpha(x)\,d\beta(y)\right| < \varepsilon$$

Since ε is arbitrary, this shows the two iterated integrals are equal. This is a case of Fubini's theorem.

26. Generalize the result of Problem 25 to the situation where α and β are only of bounded variation.

27. This problem is in Apostol [2]. Explain why whenever f is continuous on $[a,b]$

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\left(\frac{b-a}{n}\right)\right) = \int_a^b f dx.$$

Apply this to $f(x) = \frac{1}{1+x^2}$ on the interval $[0,1]$ to obtain the very interesting formula

$$\frac{\pi}{4} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}.$$

28. Suppose $f : [a,b] \times (c,d) \to \mathbb{R}$ is continuous. Recall the meaning of the partial derivative from calculus,

$$\frac{\partial f}{\partial x}(t,x) \equiv \lim_{h\to 0} \frac{f(t, x+h) - f(t,x)}{h}$$

Suppose also

$$\frac{\partial f}{\partial x}(t,x)$$

exists and for some K independent of t,

$$\left|\frac{\partial f}{\partial x}(t,z) - \frac{\partial f}{\partial x}(t,x)\right| < K\,|z - x|\,.$$

This last condition happens, for example if $\frac{\partial^2 f(t,x)}{\partial x^2}$ is uniformly bounded on $[a,b] \times (c,d)$. (Why?) Define

$$F(x) \equiv \int_a^b f(t,x)\,dt.$$

Take the difference quotient of F and show using the mean value theorem and the above assumptions that

$$F'(x) = \int_a^b \frac{\partial f(t,x)}{\partial x} dt.$$

Is there a version of this result with dt replaced with $d\alpha$ where α is an increasing function? How about α a function of bounded variation?

29. I found this problem in Apostol's book [2]. This is a very important result and is obtained very simply. Let

$$g(x) \equiv \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$$

and

$$f(x) \equiv \left(\int_0^x e^{-t^2} dt \right)^2.$$

Note

$$\frac{\partial}{\partial x} \left(\frac{e^{-x^2(1+t^2)}}{1+t^2} \right) = -2xe^{-x^2(1+t^2)}$$

and

$$\frac{\partial^2}{\partial x^2} \left(\frac{e^{-x^2(1+t^2)}}{1+t^2} \right) = -2e^{-x^2(1+t^2)} + 4x^2 e^{-x^2(1+t^2)} + 4x^2 e^{-x^2(1+t^2)} t^2$$

which is bounded for $t \in [0,1]$ and $x \in (-\infty, \infty)$. Explain why this is so. Also show the conditions of Problem 28 are satisfied so that

$$g'(x) = \int_0^1 \left(-2xe^{-x^2(1+t^2)} \right) dt.$$

Now use the chain rule and the fundamental theorem of calculus to find $f'(x)$. Then change the variable in the formula for $f'(x)$ to make it an integral from 0 to 1 and show

$$f'(x) + g'(x) = 0.$$

Now this shows $f(x) + g(x)$ is a constant. Show the constant is $\pi/4$ by assigning $x = 0$. Next take a limit as $x \to \infty$ to obtain the following formula for the improper integral, $\int_0^\infty e^{-t^2} dt$,

$$\left(\int_0^\infty e^{-t^2} dt \right)^2 = \pi/4.$$

In passing to the limit in the integral for g as $x \to \infty$ you need to justify why that integral converges to 0. To do this, argue the integrand converges uniformly to 0 for $t \in [0,1]$ and then explain why this gives convergence of the integral. Thus

$$\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/4.$$

30. The gamma function is defined for $x > 0$ as

$$\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1} dt \equiv \lim_{R \to \infty} \int_0^R e^{-t} t^{x-1} dt$$

Show this limit exists. Note you might have to give a meaning to

$$\int_0^R e^{-t} t^{x-1} dt$$

if $x < 1$. Also show that

$$\Gamma(x+1) = x\Gamma(x), \; \Gamma(1) = 1.$$

How does $\Gamma(n)$ for n an integer compare with $(n-1)!$?

31. Find $\Gamma\left(\frac{1}{2}\right)$. **Hint:** $\Gamma\left(\frac{1}{2}\right) \equiv \int_0^\infty e^{-t} t^{-1/2} dt$. Explain carefully why this equals

$$2 \int_0^\infty e^{-u^2} du$$

Then use Problem 29. Find a formula for $\Gamma\left(\frac{3}{2}\right), \Gamma\left(\frac{5}{2}\right)$, etc.

32. For $p, q > 0$, $B(p,q) \equiv \int_0^1 x^{p-1}(1-x)^{q-1} dx$. This is called the beta function. Show $\Gamma(p)\Gamma(q) = B(p,q)\Gamma(p+q)$. **Hint:** You might want to adapt and use the Fubini theorem presented earlier in Problem 25 about iterated integrals.

Fourier Series

10.1 The Complex Exponential

What does e^{ix} mean? Here $i^2 = -1$. Recall the complex numbers are of the form $a + ib$ and are identified as points in the plane. For $f(x) = e^{ix}$, you would want

$$f''(x) = i^2 f(x) = -f(x)$$

so

$$f''(x) + f(x) = 0.$$

Also, you would want

$$f(0) = e^0 = 1,\ f'(0) = ie^0 = i.$$

One solution to these conditions is

$$f(x) = \cos(x) + i\sin(x).$$

Is it the only solution? Suppose $g(x)$ is another solution. Consider $u(x) = f(x) - g(x)$. Then it follows

$$u''(x) + u(x) = 0,\ u(0) = 0 = u'(0).$$

Also, both $\operatorname{Re} u$ and $\operatorname{Im} u$ satisfy this equation. Therefore, by Lemma 8.3.10 both $\operatorname{Re} u$ and $\operatorname{Im} u$ are equal to 0. Thus the above is the only solution. Recall by De'Moivre's theorem

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$$

for any integer n and so

$$\left(e^{ix}\right)^n = e^{inx}.$$

10.2 Definition And Basic Properties

A Fourier series is an expression of the form

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

where this means

$$\lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}.$$

Obviously such a sequence of partial sums may or may not converge at a particular value of x.

These series have been important in applied math since the time of Fourier who was an officer in Napoleon's army. He was interested in studying the flow of heat in cannons and invented the concept to aid him in his study. Since that time, Fourier series and the mathematical problems related to their convergence have motivated the development of modern methods in analysis. As recently as the mid 1960's a problem related to convergence of Fourier series was solved for the first time and the solution of this problem was a big surprise.[1] This chapter is on the classical theory of convergence of Fourier series.

If you can approximate a function f with an expression of the form

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

then the function must have the property $f(x+2\pi) = f(x)$ because this is true of every term in the above series. More generally, here is a definition.

Definition 10.2.1 *A function, f defined on $\mathbb{R}$ is a periodic function of period T if $f(x+T) = f(x)$ for all x.*

As just explained, Fourier series are useful for representing periodic functions and no other kind of function.There is no loss of generality in studying only functions which are periodic of period 2π. Indeed, if f is a function which has period T, you can study this function in terms of the function, $g(x) \equiv f\left(\frac{Tx}{2\pi}\right)$ where g is periodic of period 2π.

Definition 10.2.2 *For $f \in R([-\pi,\pi])$ and f periodic on $\mathbb{R}$, define the Fourier series of f as*

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}, \tag{10.1}$$

where

$$c_k \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-iky} dy. \tag{10.2}$$

Also define the n^{th} partial sum of the Fourier series of f by

$$S_n(f)(x) \equiv \sum_{k=-n}^{n} c_k e^{ikx}. \tag{10.3}$$

It may be interesting to see where this formula came from. Suppose then that

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

multiply both sides by e^{-imx} and take the integral $\int_{-\pi}^{\pi}$, so that

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx} dx = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} c_k e^{ikx} e^{-imx} dx.$$

[1] The question was whether the Fourier series of a function in L^2 converged a.e. to the function. It turned out that it did, to the surprise of many because it was known that the Fourier series of a function in L^1 does not necessarily converge to the function a.e. The problem was solved by Carleson in 1965.

Now switch the sum and the integral on the right side even though there is absolutely no reason to believe this makes any sense. Then

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) e^{-imx} dx &= \sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx \\ &= c_m \int_{-\pi}^{\pi} 1 dx = 2\pi c_m\end{aligned}$$

because $\int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx = 0$ if $k \neq m$. It is formal manipulations of the sort just presented which suggest that Definition 10.2.2 might be interesting.

In case f is real valued, $\overline{c_k} = c_{-k}$ and so

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) dy + \sum_{k=1}^{n} 2 \operatorname{Re}\left(c_k e^{ikx}\right).$$

Letting $c_k \equiv \alpha_k + i\beta_k$

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) dy + \sum_{k=1}^{n} 2 \left[\alpha_k \cos kx - \beta_k \sin kx\right]$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) (\cos ky - i \sin ky) dy$$

which shows that

$$\alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \cos(ky) dy, \ \beta_k = \frac{-1}{2\pi} \int_{-\pi}^{\pi} f(y) \sin(ky) dy$$

Therefore, letting $a_k = 2\alpha_k$ and $b_k = -2\beta_k$,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos(ky) dy, \ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin(ky) dy$$

and

$$S_n f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \tag{10.4}$$

This is often the way Fourier series are presented in elementary courses where it is only real functions which are to be approximated. However it is easier to stick with Definition 10.2.2.

The partial sums of a Fourier series can be written in a particularly simple form which is presented next.

$$\begin{aligned}S_n f(x) &= \sum_{k=-n}^{n} c_k e^{ikx} \\ &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy\right) e^{ikx} \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{k=-n}^{n} \left(e^{ik(x-y)}\right) f(y) dy \\ &\equiv \int_{-\pi}^{\pi} D_n(x-y) f(y) dy. \end{aligned} \tag{10.5}$$

The function,

$$D_n(t) \equiv \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikt}$$

is called the Dirichlet Kernel

Theorem 10.2.3 *The function, D_n satisfies the following:*

1. $\int_{-\pi}^{\pi} D_n(t)\,dt = 1$
2. D_n *is periodic of period* 2π
3. $D_n(t) = (2\pi)^{-1}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}$.

Proof: Part 1 is obvious because $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iky}dy = 0$ whenever $k \neq 0$ and it equals 1 if $k = 0$. Part 2 is also obvious because $t \to e^{ikt}$ is periodic of period 2π. It remains to verify Part 3. Note

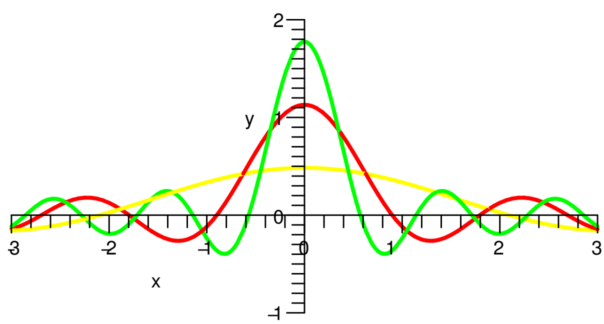
$$2\pi D_n(t) = \sum_{k=-n}^{n} e^{ikt} = 1 + 2\sum_{k=1}^{n}\cos(kt)$$

Therefore,

$$\begin{aligned}
2\pi D_n(t)\sin\left(\frac{t}{2}\right) &= \sin\left(\frac{t}{2}\right) + 2\sum_{k=1}^{n}\sin\left(\frac{t}{2}\right)\cos(kt) \\
&= \sin\left(\frac{t}{2}\right) + \sum_{k=1}^{n}\sin\left(\left(k+\frac{1}{2}\right)t\right) - \sin\left(\left(k+\frac{1}{2}\right)t\right) \\
&= \sin\left(\left(n+\frac{1}{2}\right)t\right)
\end{aligned}$$

where the easily verified trig. identity $\cos(a)\sin(b) = \sin(a+b) - \sin(a-b)$ is used to get to the second line. This proves 3 and proves the theorem.

Here is a picture of the Dirichlet kernels for n=1,2, and 3



Note they are not nonnegative but there is a large central positive bump which gets larger as n gets larger.

It is not reasonable to expect a Fourier series to converge to the function at every point. To see this, change the value of the function at a single point in $(-\pi, \pi)$ and extend to keep the modified function periodic. Then the Fourier series of the modified function is the same as the Fourier series of the original function and so if pointwise convergence did take place, it no longer does. However, it is possible to prove an interesting theorem about pointwise convergence of Fourier series. This is done next.

10.3 The Riemann Lebesgue Lemma

The Riemann Lebesgue lemma is the basic result which makes possible the study of pointwise convergence of Fourier series. It is also a major result in other contexts and serves as a useful example.

For the purpose of simple notation, let $R((a,b])$ denote those functions f which are in $R([a+\delta,b])$ for every $\delta>0$ and the improper integral

$$\lim_{\delta\to 0}\int_{a+\delta}^{b} f(x)\,dx \text{ exists.}$$

Lemma 10.3.1 *Let $f\in R((a,b]), f(x)\geq 0,$ where (a,b) is some finite interval and let $\varepsilon>0$. Then there exists an interval of finite length, $[a_1,b_1]\subseteq(a,b)$ and a differentiable function, h having continuous derivative such that both h and h' equal 0 outside of $[a_1,b_1]$ which has the property that $0\leq h(x)\leq f(x)$ and*

$$\int_a^b |f-h|\,dx<\varepsilon \tag{10.6}$$

Proof: First here is a claim.

Claim: There exists a continuous g which vanishes near a and b such that $g\leq f$ and

$$\int_a^b |f(x)-g(x)|\,dx<\varepsilon/3.$$

where the integral is the improper Riemann integral defined by

$$\int_a^b (f(x)-g(x))\,dx\equiv\lim_{\delta\to 0+}\int_{a+\delta}^{b}(f(x)-g(x))\,dx$$

Proof of the claim: First let $a_0>a$ such that

$$\left|\int_a^b f(x)\,dx-\int_{a_0}^b f(x)\,dx\right|=\int_a^{a_0}|f(x)|\,dx<\varepsilon/3.$$

Let $\{x_0,x_1,\cdots,x_n\}$ be a partition of $[a_0,b]$ and let

$$\sum_{k=1}^{n}a_k(x_k-x_{k-1})$$

be a lower sum such that

$$\left|\int_{a_0}^b f(x)\,dx-\sum_{k=1}^{n}a_k(x_k-x_{k-1})\right|<\varepsilon/6.$$

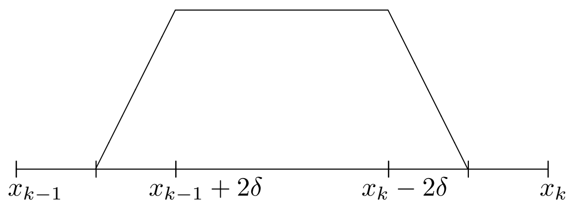
The sum in the above equals

$$\int_{a_0}^{b}\sum_{k=1}^{n}a_k\mathcal{X}_{[x_{k-1},x_k)}(x)\,dx$$

where

$$\mathcal{X}_{[x_{k-1},x_k)}(x) = \begin{cases} 1 \text{ if } x \in [x_{k-1}, x_k) \\ 0 \text{ if } x \notin [x_{k-1}, x_k) \end{cases}$$

Now let ψ_k be a continuous function which approximates $\mathcal{X}_{[x_{k-1},x_k)}(x)$ and vanishes near the endpoints of $[x_{k-1}, x_k)$ as shown in the following picture in which δ is sufficiently small.



Then

$$\begin{aligned}
&\int_{a_0}^{b} \left|\mathcal{X}_{[x_{k-1},x_k)}(x) - \psi_k(x)\right| a_k dx \\
\leq\ & \int_{x_{k-1}}^{x_{k-1}+\delta} a_k dx + \int_{x_{k-1}+\delta}^{x_{k-1}+2\delta} a_k \left|\mathcal{X}_{[x_{k-1},x_k)}(x) - \psi_k(x)\right| dx \\
&+ \int_{x_k-2\delta}^{x_k-\delta} a_k \left|\mathcal{X}_{[x_{k-1},x_k)}(x) - \psi_k(x)\right| dx + \int_{x_k-\delta}^{x_k} a_k \\
&\leq \delta a_k + \delta a_k + \delta a_k + \delta a_k = 4\delta a_k.
\end{aligned}$$

Let δ be chosen small enough that the above expression is less than $\varepsilon/6n$. Therefore, choosing ψ_k in this manner for each $k = 1, \cdot\cdot\cdot, n$ yields

$$\begin{aligned}
&\left|\int_{a_0}^{b} \sum_{k=1}^{n} a_k \mathcal{X}_{[x_{k-1},x_k)}(x)\, dx - \int_{a_0}^{b} \sum_{k=1}^{n} a_k \psi_k(x)\, dx\right| \\
=\ & \int_{a_0}^{b} \sum_{k=1}^{n} a_k \left|\mathcal{X}_{[x_{k-1},x_k)}(x) - \psi_k(x)\right| dx < n \times \frac{\varepsilon}{6n} = \varepsilon/6.
\end{aligned}$$

Let $g(x) = \sum_{k=1}^{n} a_k \psi_k(x)$.

$$\begin{aligned}
\int_{a_0}^{b} |f(x) - g(x)|\, dx &= \left|\int_{a_0}^{b} (f(x) - g(x))\, dx\right| \\
&\leq \left|\int_{a_0}^{b} f(x)\, dx - \sum_{k=1}^{n} a_k (x_k - x_{k-1})\right| \\
&+ \left|\sum_{k=1}^{n} a_k (x_k - x_{k-1}) - \int_{a_0}^{b} \sum_{k=1}^{n} a_k \psi_k(x)\, dx\right| < \varepsilon/6 + \varepsilon/6 = \varepsilon/3.
\end{aligned}$$

This proves the claim.

Now say g equals zero off $[a', b'] \subseteq (a, b)$. Then for small $h > 0$,

$$h < \min\left((a' - a)/3, (b - b')/3\right) \tag{10.7}$$

define

$$g_h(x) \equiv \frac{1}{2h} \int_{x-h}^{x+h} g(t)\, dt$$

Then g_h is continuous and has a continuous derivative which equals

$$\frac{1}{2h}\left(g\left(x+h\right)-g\left(x-h\right)\right).$$

Say $x\in[x_{k-1},x_k)$ and let δ be as above. Let $h<\delta/2$. Then $g_h\left(x\right)\leq f\left(x\right)$ because on this interval $f\left(x\right)\geq a_k$ and $g\left(x\right)\leq a_k$. Also

$$\int_{a_0}^{b}\left|g\left(x\right)-g_h\left(x\right)\right|dx=\int_{a_0}^{b}\left|g\left(x\right)-\frac{1}{2h}\int_{x-h}^{x+h}g\left(t\right)dt\right|dx$$

$$=\int_{a_0}^{b}\left|\frac{1}{2h}\int_{x-h}^{x+h}\left(g\left(x\right)-g\left(t\right)\right)dt\right|dx\leq\int_{a_0}^{b}\frac{1}{2h}\int_{x-h}^{x+h}\left|g\left(x\right)-g\left(t\right)\right|dtdx$$

$$<\int_{a_0}^{b}\frac{1}{2h}\int_{x-h}^{x+h}\frac{\varepsilon}{3\left(b-a_0\right)}dtdx=\frac{\varepsilon}{3}$$

provided h is small enough, due to the uniform continuity of g. (Why is g uniformly continuous?) Also, since h satisfies 10.7, g_h and g_h' vanish outside some closed interval, $[a_1,b_1]\subseteq(a_0,b)$. Since g_h equals zero between a and a_0, this shows that for such h,

$$\int_{a}^{b}\left|f\left(x\right)-g_h\left(x\right)\right|dx=\left|\int_{a}^{b}f\left(x\right)dx-\int_{a}^{b}g_h\left(x\right)dx\right|$$

$$\begin{aligned}
&\leq\left|\int_{a}^{b}f\left(x\right)dx-\int_{a_0}^{b}f\left(x\right)dx\right|+\left|\int_{a_0}^{b}f\left(x\right)dx-\int_{a_0}^{b}g_h\left(x\right)dx\right|\\
&\leq\frac{\varepsilon}{3}+\left|\int_{a_0}^{b}f\left(x\right)dx-\int_{a_0}^{b}g\left(x\right)dx\right|+\left|\int_{a_0}^{b}g\left(x\right)dx-\int_{a_0}^{b}g_h\left(x\right)dx\right|\\
&<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.
\end{aligned}$$

Letting $h=g_h$ this proves the Lemma.

The lemma can be generalized to the case where f has values in $\mathbb{C}$. In this case,

$$\int_{a}^{b}f\left(x\right)dx\equiv\int_{a}^{b}\operatorname{Re}f\left(x\right)dx+i\int_{a}^{b}\operatorname{Im}f\left(x\right)dx$$

and $f\in R\left((a,b]\right)$ means $\operatorname{Re}f,\operatorname{Im}f\in R\left((a,b]\right)$.

Lemma 10.3.2 *Let $|f|\in R\left((a,b]\right)$, where (a,b) is some finite interval and let $\varepsilon>0$. Then there exists an interval of finite length, $[a_1,b_1]\subseteq(a,b)$ and a differentiable function, h having continuous derivative such that both h and h' equal 0 outside of $[a_1,b_1]$ which has the property that*

$$\int_{a}^{b}\left|f-h\right|dx<\varepsilon \tag{10.8}$$

Proof: For g a real valued bounded function, define

$$g_+\left(x\right)\equiv\frac{g\left(x\right)+\left|g\left(x\right)\right|}{2},\ g_-\left(x\right)\equiv\frac{\left|g\left(x\right)\right|-g\left(x\right)}{2}.$$

First suppose f is real valued. Then apply Lemma 10.3.2 to f_+ and f_- to obtain h_+ and h_- continuous and vanishing on some interval, $[a_1, b_1] \subseteq (a, b)$ such that

$$\int_a^b (f_+ - h_+)\, dx < \varepsilon/5, \int_a^b (f_- - h_-)\, dx < \varepsilon/5$$

Let $h = h_+ - h_-$. Then if $f(x) \geq 0, f_-(x) = 0$ and so

$$f(x) - h(x) = f_+(x) - h_+(x) + h_-(x) = f_+(x) - h_+(x) \geq 0$$

If $f(x) < 0$, then $f_+(x) = 0$ and so $h_+(x) = 0$ and

$$h(x) - f(x) = -h_-(x) - (-f_-(x)) = f_-(x) - h_-(x) \geq 0.$$

Therefore, $|f(x) - h(x)| \leq f_+(x) - h_+(x) + (f_-(x) - h_-(x))$ and so

$$\int_a^b |f - h|\, dx \leq \int_a^b (f_+ - h_+)\, dx + \int_a^b (f_- - h_-)\, dx < \frac{2\varepsilon}{5}.$$

Now if f has values in $\mathbb{C}$, from what was just shown, there exist h_1, h_2 continuous and vanishing on some interval, $[a_1, b_1] \subseteq (a, b)$ such that

$$\int_a^b |\mathrm{Re}\, f - h_1|\, dx, \int_a^b |\mathrm{Im}\, f - h_2|\, dx < \frac{2\varepsilon}{5}$$

and therefore,

$$\int_a^b |\mathrm{Re}\, f + i\,\mathrm{Im}\, f - (h_1 + ih_2)|\, dx \leq \int_a^b |\mathrm{Re}\, f - h_1|\, dx$$

$$+ \int_a^b |\mathrm{Im}\, f - h_2|\, dx < \frac{4\varepsilon}{5} < \varepsilon.$$

This proves the lemma.

The lemma is the basis for the Riemann Lebesgue lemma, the main result in the study of pointwise convergence of Fourier series.

Lemma 10.3.3 *(Riemann Lebesgue) Let $|f| \in R((a, b])$ where (a, b) is some finite interval. Then*

$$\lim_{\alpha \to \infty} \int_a^b f(t) \sin(\alpha t + \beta)\, dt = 0. \tag{10.9}$$

Here the integral is the improper Riemann integral defined by

$$\lim_{\delta \to 0+} \int_{a+\delta}^b f(t) \sin(\alpha t + \beta)\, dt$$

Proof: Let $\varepsilon > 0$ be given and use Lemma 10.3.2 to obtain g such that g and g' are both continuous and vanish outside $[a_1, b_1] \subseteq (a, b)$, and

$$\int_a^b |g - f|\, dx < \frac{\varepsilon}{2}. \tag{10.10}$$

Then

$$\left| \int_a^b f(t) \sin(\alpha t + \beta)\, dt \right| \leq$$

$$\begin{aligned}
& \left|\int_a^b (f(t) - g(t)) \sin(\alpha t + \beta)\, dt\right| \\
& + \left|\int_a^b g(t) \sin(\alpha t + \beta)\, dt\right| \\
\leq\ & \int_a^b |f - g|\, dx + \left|\int_a^b g(t) \sin(\alpha t + \beta)\, dt\right| \\
<\ & \frac{\varepsilon}{2} + \left|\int_{a_1}^{b_1} g(t) \sin(\alpha t + \beta)\, dt\right|.
\end{aligned}$$

Integrate the last term by parts.

$$\int_{a_1}^{b_1} g(t) \sin(\alpha t + \beta)\, dt = \frac{-\cos(\alpha t + \beta)}{\alpha} g(t) |_{a_1}^{b_1} + \int_{a_1}^{b_1} \frac{\cos(\alpha t + \beta)}{\alpha} g'(t)\, dt,$$

an expression which converges to zero since g' is bounded and

$$\frac{-\cos(\alpha t + \beta)}{\alpha} g(t) |_{a_1}^{b_1} = 0$$

because g vanishes at a_1 and b_1. Therefore, taking α large enough,

$$\left|\int_a^b f(t) \sin(\alpha t + \beta)\, dt\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and this proves the lemma.

10.4 Dini's Criterion For Convergence

Fourier series like to converge to the midpoint of the jump of a function under certain conditions. The condition given for convergence in the following theorem is due to Dini. It is a generalization of the usual theorem presented in elementary books on Fourier series methods. [3].

Recall

$$\lim_{t \to x+} f(t) \equiv f(x+), \text{ and } \lim_{t \to x-} f(t) \equiv f(x-)$$

Theorem 10.4.1 *Let f be a periodic function of period 2π which is in $R([-\pi, \pi])$. Suppose at some x, $f(x+)$ and $f(x-)$ both exist and that the function*

$$y \to \left|\frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y}\right| \equiv h(y) \tag{10.11}$$

is in $R((0, \pi])$ which means

$$\lim_{\varepsilon \to 0+} \int_\varepsilon^\pi h(y)\, dy \text{ exists} \tag{10.12}$$

Then

$$\lim_{n \to \infty} S_n f(x) = \frac{f(x+) + f(x-)}{2}. \tag{10.13}$$

Proof:

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y)\, dy$$

Change variables $x - y \to y$ and use the periodicity of f and D_n along with the formula for $D_n(y)$ to write this as

$$\begin{aligned} S_n f(x) &= \int_{-\pi}^{\pi} D_n(y) f(x-y) \\ &= \int_{0}^{\pi} D_n(y) f(x-y)\, dy + \int_{-\pi}^{0} D_n(y) f(x-y)\, dy \\ &= \int_{0}^{\pi} D_n(y) [f(x-y) + f(x+y)]\, dy \\ &= \int_{0}^{\pi} \frac{1}{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right) y\right)}{\sin\left(\frac{y}{2}\right)} \left[\frac{f(x-y) + f(x+y)}{2}\right] dy. \end{aligned} \tag{10.14}$$

Note the function

$$y \to \frac{1}{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right) y\right)}{\sin\left(\frac{y}{2}\right)},$$

while it is not defined at 0, is at least bounded and by L'Hospital's rule,

$$\lim_{y \to 0} \frac{1}{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right) y\right)}{\sin\left(\frac{y}{2}\right)} = \frac{2n+1}{\pi}$$

so defining it to equal this value at 0 yields a continuous, hence Riemann integrable function and so the above integral at least makes sense. Also from the property that $\int_{-\pi}^{\pi} D_n(t)\, dt = 1$,

$$\begin{aligned} f(x+) + f(x-) &= \int_{-\pi}^{\pi} D_n(y) [f(x+) + f(x-)]\, dy \\ &= 2 \int_{0}^{\pi} D_n(y) [f(x+) + f(x-)]\, dy \\ &= \int_{0}^{\pi} \frac{1}{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right) y\right)}{\sin\left(\frac{y}{2}\right)} [f(x+) + f(x-)]\, dy \end{aligned}$$

and so

$$\left| S_n f(x) - \frac{f(x+) + f(x-)}{2} \right| =$$

$$\left| \int_{0}^{\pi} \frac{1}{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right) y\right)}{\sin\left(\frac{y}{2}\right)} \left[\frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2}\right] dy \right|. \tag{10.15}$$

Now the function

$$y \to \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2 \sin\left(\frac{y}{2}\right)} \right| \tag{10.16}$$

satisfies the condition 10.12. To see this, note the numerator is in $R([0, \pi])$ because f is. Therefore, this function is in $R([\delta, \pi])$ for each $\delta > 0$ because $\sin\left(\frac{y}{2}\right)$ is bounded below by $\sin\left(\frac{\delta}{2}\right)$ for such y. It remains to show it is in $R((0, \pi])$. For $\varepsilon_1 < \varepsilon_2$,

$$\int_{\varepsilon_1}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2 \sin\left(\frac{y}{2}\right)} \right| dy$$

$$- \int_{\varepsilon_2}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2 \sin\left(\frac{y}{2}\right)} \right| dy$$

$$= \int_{\varepsilon_1}^{\varepsilon_2} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2\sin\left(\frac{y}{2}\right)} \right| dy$$

$$\begin{aligned} &= \int_{\varepsilon_1}^{\varepsilon_2} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| \frac{y}{2\sin\left(\frac{y}{2}\right)} dy \\ &\leq \int_{\varepsilon_1}^{\varepsilon_2} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy \end{aligned}$$

$$\begin{aligned} = &\int_{\varepsilon_1}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy \\ &- \int_{\varepsilon_2}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy \end{aligned}$$

Letting $\{\varepsilon_k\}$ be any sequence of positive numbers converging to 0, this shows

$$\left\{ \int_{\varepsilon_k}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2\sin\left(\frac{y}{2}\right)} \right| dy \right\}_{k=1}^{\infty}$$

is a Cauchy sequence because the difference between the k^{th} and the m^{th} terms, $\varepsilon_k < \varepsilon_m$, is no larger than the difference between the k^{th} and m^{th} terms of the sequence

$$\left\{ \int_{\varepsilon_k}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy \right\}_{k=1}^{\infty}$$

which is given to be Cauchy. Since the function,

$$\varepsilon \to \int_{\varepsilon}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{2\sin\left(\frac{y}{2}\right)} \right| dy$$

is decreasing, the limit of this function must exist.

Thus the function in 10.16 is in $R((0,\pi])$ as claimed. It follows from the Riemann Lebesgue lemma, that 10.15 converges to zero as $n \to \infty$. This proves the theorem.

The following corollary is obtained immediately from the above proof with minor modifications.

Corollary 10.4.2 *Let f be a periodic function of period 2π which is an element of $R([-\pi,\pi])$. Suppose at some x, the function*

$$y \to \left| \frac{f(x-y) + f(x+y) - 2s}{y} \right| \tag{10.17}$$

is in $R((0,\pi])$. Then

$$\lim_{n\to\infty} S_n f(x) = s. \tag{10.18}$$

The following corollary gives an easy to check condition for the Fourier series to converge to the mid point of the jump.

Corollary 10.4.3 *Let f be a periodic function of period 2π which is an element of $R([-\pi,\pi])$. Suppose at some x, $f(x+)$ and $f(x-)$ both exist and there exist positive constants, K and δ such that whenever $0 < y < \delta$*

$$|f(x-y) - f(x-)| \leq K y^{\theta}, \ |f(x+y) - f(x+)| < K y^{\theta} \tag{10.19}$$

where $\theta \in (0,1]$. *Then*

$$\lim_{n\to\infty} S_n f(x) = \frac{f(x+) + f(x-)}{2}. \tag{10.20}$$

Proof: The condition 10.19 clearly implies Dini's condition, 10.11. This is because for $0 < y < \delta$

$$\left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| \leq 2Ky^{\theta-1}$$

and

$$\int_{\varepsilon}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy$$

$$\leq \int_{\varepsilon}^{\delta} 2Ky^{\theta-1} dy + \frac{1}{\delta} \int_{\delta}^{\pi} |f(x-y) - f(x-) + f(x+y) - f(x+)| \, dy.$$

Now

$$\int_{\varepsilon}^{\delta} 2Ky^{\theta-1} dy = 2K \frac{\delta^{\theta} - \varepsilon^{\theta}}{\theta}$$

which converges to

$$2K \frac{\delta^{\theta}}{\theta}$$

as $\varepsilon \to 0$. Thus

$$\lim_{\varepsilon\to 0+} \int_{\varepsilon}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy$$

exists and equals

$$\sup \left\{ \int_{\varepsilon}^{\pi} \left| \frac{f(x-y) - f(x-) + f(x+y) - f(x+)}{y} \right| dy : \varepsilon > 0 \right\}$$

because it is bounded above by

$$2K \frac{\delta^{\theta}}{\theta} + \frac{1}{\delta} \int_{\delta}^{\pi} |f(x-y) - f(x-) + f(x+y) - f(x+)| \, dy.$$

(Why?) This proves the corollary.

As pointed out by Apostol [3], where you can read more of this sort of thing, this is a very remarkable result because even though the Fourier coeficients depend on the values of the function on all of $[-\pi, \pi]$, the convergence properties depend in this theorem on very local behavior of the function.

10.5 Integrating And Differentiating Fourier Series

You can typically integrate Fourier series term by term and things will work out according to your expectations. More precisely, if the Fourier series of f is

$$\sum_{k=-\infty}^{\infty} a_k e^{ikx}$$

then it will be true for $x \in [-\pi, \pi]$ that

$$F(x) \equiv \int_{-\pi}^{x} f(t) \, dt = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k \int_{-\pi}^{x} e^{ikt} dt$$

$$= a_0 (x + \pi) + \lim_{n\to\infty} \sum_{k=-n, k\neq 0}^{n} a_k \left(\frac{e^{ikx}}{ik} - \frac{(-1)^k}{ik} \right).$$

I shall show this is true for the case where f is an arbitrary 2π periodic function which on $(-\pi, \pi)$ is the restriction of a continuous function on $[-\pi, \pi]$ but it holds for any $f \in R([-\pi, \pi])$ and in the case of a more general notion of integration, it holds for more general functions than these. Note it is not necessary to assume anything about the function, f being the limit of its Fourier series. Let

$$G(x) \equiv F(x) - a_0 (x + \pi) = \int_{-\pi}^{x} (f(t) - a_0)\, dt$$

Then G equals 0 at $-\pi$ and π. Therefore, the periodic extension of G is continuous. Also

$$|G(x) - G(x_1)| \leq \left| \int_{x_1}^{x} M dt \right| \leq M |x - x_1|$$

where M is an upper bound for $|f(t) - a_0|$. Thus the Dini condition of Corollary 10.4.3 holds. Therefore for all $x \in \mathbb{R}$,

$$G(x) = \sum_{k=-\infty}^{\infty} A_k e^{ikx} \tag{10.21}$$

where

$$A_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x) e^{ikx} dx$$

Now from 10.21 and the definition of the Fourier coefficients for f,

$$G(\pi) = F(\pi) - a_0 2\pi = 0 = A_0 + \lim_{n\to\infty} \sum_{k=-n, k\neq 0}^{n} A_k (-1)^k \tag{10.22}$$

Next consider A_k for $k \neq 0$.

$$A_k \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} G(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{x} (f(t) - a_0)\, dt e^{-ikx} dx$$

and now this is integrated by parts using the fundamental theorem of calculus. This is the only place where f is continuous is used. (There are other arguments using, for example, Fubini's theorem which could be applied at this stage which require very little about f. However, this mathematical machinery has not been discussed.)

$$A_k = \frac{1}{2\pi} \left(-\frac{e^{-ikx}}{ik} \right) \int_{-\pi}^{x} (f(t) - a_0)\, dt |_{-\pi}^{\pi} + \frac{1}{2\pi i k} \int_{-\pi}^{\pi} (f(x) - a_0) e^{-ikx} dx$$

Now from the definition of a_0,

$$\int_{-\pi}^{\pi} (f(t) - a_0)\, dt = 0$$

and so

$$A_k = \frac{1}{2\pi i k} \int_{-\pi}^{\pi} (f(x) - a_0) e^{-ikx} dx = \frac{a_k}{ik}.$$

From 10.21 and 10.22

$$F(x) - a_0 (x + \pi) = \lim_{n\to\infty} \sum_{k=-n, k\neq 0}^{n} \frac{a_k}{ik} e^{ikx} - \sum_{k=-n, k\neq 0}^{n} \frac{a_k}{ik} (-1)^k$$

and so

$$\begin{aligned} F(x) &= \int_{-\pi}^{x} f(t)\,dt = \int_{-\pi}^{x} a_0 dt + \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} \frac{a_k}{ik}\left(e^{ikx} - (-1)^k\right) \\ &= \int_{-\pi}^{x} a_0 dt + \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} a_k \int_{-\pi}^{x} e^{ikt} dt \end{aligned}$$

This proves the following theorem.

Theorem 10.5.1 *Let f be 2π periodic and on $(-\pi,\pi)$ f is the restriction of a function continuous on $[-\pi,\pi]$. Then*

$$\int_{-\pi}^{x} f(t)\,dt = \int_{-\pi}^{x} a_0 dt + \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} a_k \int_{-\pi}^{x} e^{ikt} dt$$

where a_k are the Fourier coefficients of f.

Example 10.5.2 *Let $f(x) = x$ for $x \in [-\pi,\pi)$ and extend f to make it 2π periodic. Then the Fourier coefficients of f are*

$$a_0 = 0,\ a_k = \frac{(-1)^k i}{k}$$

Therefore, $\frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-ikt} = \frac{i}{k}\cos \pi k$

$$\int_{-\pi}^{x} tdt = \frac{1}{2}x^2 - \frac{1}{2}\pi^2$$

$$\begin{aligned} &= \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} \frac{(-1)^k i}{k} \int_{-\pi}^{x} e^{ikt} dt \\ &= \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} \frac{(-1)^k i}{k} \left(\frac{\sin xk}{k} + i\frac{-\cos xk + (-1)^k}{k}\right) \end{aligned}$$

For fun, let $x = 0$ and conclude

$$\begin{aligned} -\frac{1}{2}\pi^2 &= \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} \frac{(-1)^k i}{k}\left(i\frac{-1+(-1)^k}{k}\right) \\ &= \lim_{n\to\infty} \sum_{k=-n,k\neq 0}^{n} \frac{(-1)^{k+1}}{k}\left(\frac{-1+(-1)^k}{k}\right) \\ &= \lim_{n\to\infty} 2\sum_{k=1}^{n} \frac{(-1)^k + (-1)}{k^2} = \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2} \end{aligned}$$

and so

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

The above theorem can easily be generalized to piecewise continuous functions defined below.

Definition 10.5.3 *Let f be a function defined on $[a,b]$. It is called piecewise continuous if there is a partition of $[a,b]$, $\{x_0,\cdots,x_n\}$ such that on $[x_{k-1},x_k]$ there is a continuous function g_k such that $f(x)=g_k(x)$ for all $x\in(x_{k-1},x_k)$.*

Then the proof of the above theorem generalizes right away to yield the following corollary. It involves splitting the integral into a sum of integrals taken over the subintervals determined by the partition in the above definition and on each of these one uses a similar argument.

Corollary 10.5.4 *Let f be 2π periodic and piecewise continuous on $[-\pi,\pi]$. Then*

$$\int_{-\pi}^{x} f(t)\,dt=\int_{-\pi}^{x} a_0 dt+\lim_{n\to\infty}\sum_{k=-n,k\neq 0}^{n} a_k\int_{-\pi}^{x} e^{ikt}dt$$

where a_k are the Fourier coefficients of f.

Of course it is not reasonable to suppose you can differentiate a Fourier series term by term and get good results.

Consider the series for $f(x)=1$ if $x\in(0,\pi]$ and $f(x)=-1$ on $(-\pi,0)$ with $f(0)=0$. In this case $a_0=0$.

$$a_k=\frac{1}{2\pi}\left(\int_0^{\pi} e^{-ikt}dt-\int_{-\pi}^{0} e^{-ikt}dt\right)=\frac{i}{\pi}\frac{\cos\pi k-1}{k}$$

so the Fourier series is

$$\sum_{k\neq 0}\left(\frac{(-1)^k-1}{\pi k}\right)ie^{ikx}$$

What happens if you differentiate it term by term? It gives

$$\sum_{k\neq 0}-\frac{(-1)^k-1}{\pi}e^{ikx}$$

which fails to converge anywhere because the k^{th} term fails to converge to 0. This is in spite of the fact that f has a derivative away from 0.

However, it is possible to prove some theorems which let you differentiate a Fourier series term by term. Here is one such theorem.

Theorem 10.5.5 *Suppose for $x\in[-\pi,\pi]$*

$$f(x)=\int_{-\pi}^{x} f'(t)\,dt+f(-\pi)$$

and $f'(t)$ is piecewise continuous. Then if

$$f(x)=\sum_{k=-\infty}^{\infty} a_k e^{ikx}$$

it follows the Fourier series of f' is

$$\sum_{k=-\infty}^{\infty} a_k ike^{ikx}.$$

Proof: Since f' is piecewise continuous, 2π periodic (why?) it follows from Corollary 10.5.4

$$f(x) - f(-\pi) = \sum_{k=-\infty}^{\infty} b_k \left(\int_{-\pi}^{x} e^{ikt} dt \right)$$

where b_k is the k^{th} Fourier coefficient of f'. Thus

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(t) e^{-ikt} dt$$

Breaking the integral into pieces if necessary, and integrating these by parts yields finally

$$\begin{aligned} &= \frac{1}{2\pi} \left[f(t) e^{-ikt} |_{-\pi}^{\pi} + ik \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right] \\ &= ik \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt = ika_k \end{aligned}$$

where a_k is the Fourier coefficient of f. Since f is periodic of period 2π, the boundary term vanishes. It follows the Fourier series for f' is

$$\sum_{k=-\infty}^{\infty} ika_k e^{ikx}$$

as claimed. This proves the theorem.

Note the conclusion of this theorem is only about the Fourier series of f'. It does not say the Fourier series of f' converges pointwise to f'. However, if f' satisfies a Dini condition, then this will also occur. For example, if f' has a bounded derivative at every point, then by the mean value theorem $|f'(x) - f'(y)| \leq K |x - y|$ and this is enough to show the Fourier series converges to $f'(x)$ thanks to Corollary 10.4.3.

10.6 Ways Of Approximating Functions

Given above is a theorem about Fourier series converging pointwise to a periodic function or more generally to the mid point of the jump of the function. Notice that some sort of smoothness of the function approximated was required, the Dini condition. It can be shown that if this sort of thing is not present, the Fourier series of a continuous periodic function may fail to converge to it in a very spectacular manner. In fact, Fourier series don't do very well at converging pointwise. However, there is another way of converging at which Fourier series cannot be beat. It is mean square convergence.

Definition 10.6.1 *Let f be a function defined on an interval, $[a, b]$. Then a sequence, $\{g_n\}$ of functions is said to converge uniformly to f on $[a, b]$ if*

$$\lim_{n \to \infty} \sup \{ |f(x) - g_n(x)| : x \in [a, b] \} = 0.$$

The expression $\sup \{ |f(x) - g_n(x)| : x \in [a, b] \}$ *is sometimes written*[2] *as*

$$||f - g_n||_0$$

[2]There is absolutely no consistency in this notation. It is often the case that $||\cdot||_0$ is what is referred to in this definition as $||\cdot||_2$. Also $||\cdot||_0$ here is sometimes referred to as $||\cdot||_\infty$. Sometimes $||\cdot||_2$ referrs to a norm which involves derivatives of the function.

More generally, if f is a function,

$$||f||_0 \equiv \sup\{|f(x)| : x \in [a,b]\}$$

The sequence is said to converge mean square to f if

$$\lim_{n\to\infty} ||f - g_n||_2 \equiv \lim_{n\to\infty} \left(\int_a^b |f - g_n|^2 \, dx\right)^{1/2} = 0$$

10.6.1 Uniform Approximation With Trig. Polynomials

It turns out that if you don't insist the a_k be the Fourier coefficients, then every continuous 2π periodic function $\theta \to f(\theta)$ can be approximated uniformly with a Trig. polynomial of the form

$$p_n(\theta) \equiv \sum_{k=-n}^{n} a_k e^{ik\theta}$$

This means that for all $\varepsilon > 0$ there exists a $p_n(\theta)$ such that

$$||f - p_n||_0 < \varepsilon.$$

Definition 10.6.2 *Recall the n^{th} partial sum of the Fourier series $S_n f(x)$ is given by*

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y) \, dy = \int_{-\pi}^{\pi} D_n(t) f(x-t) \, dt$$

where $D_n(t)$ is the Dirichlet kernel,

$$D_n(t) = (2\pi)^{-1} \frac{\sin\left(n + \frac{1}{2}\right) t}{\sin\left(\frac{t}{2}\right)}$$

The n^{th} Fejer mean, $\sigma_n f(x)$ is the average of the first n of the $S_n f(x)$. Thus

$$\sigma_{n+1} f(x) \equiv \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) = \int_{-\pi}^{\pi} \left(\frac{1}{n+1} \sum_{k=0}^{n} D_k(t)\right) f(x-t) \, dt$$

The Fejer kernel is

$$F_{n+1}(t) \equiv \frac{1}{n+1} \sum_{k=0}^{n} D_k(t).$$

As was the case with the Dirichlet kernel, the Fejer kernel has some properties.

Lemma 10.6.3 *The Fejer kernel has the following properties.*

1. $F_{n+1}(t) = F_{n+1}(t + 2\pi)$
2. $\int_{-\pi}^{\pi} F_{n+1}(t) \, dt = 1$
3. $\int_{-\pi}^{\pi} F_{n+1}(t) f(x-t) \, dt = \sum_{k=-n}^{n} b_k e^{ik\theta}$ *for a suitable choice of b_k.*
4. $F_{n+1}(t) = \frac{1 - \cos((n+1)t)}{4\pi(n+1)\sin^2\left(\frac{t}{2}\right)}$, $F_{n+1}(t) \geq 0, F_n(t) = F_n(-t)$.

5. *For every* $\delta > 0$,

$$\lim_{n\to\infty} \sup \{F_{n+1}(t) : \pi \geq |t| \geq \delta\} = 0.$$

In fact, for $|t| \geq \delta$,

$$F_{n+1}(t) \leq \frac{2}{(n+1)\sin^2\left(\frac{\delta}{2}\right) 4\pi}.$$

Proof: Part 1.) is obvious because F_{n+1} is the average of functions for which this is true.

Part 2.) is also obvious for the same reason as Part 1.). Part 3.) is obvious because it is true for D_n in place of F_{n+1} and then taking the average yields the same sort of sum.

The last statements in 4.) are obvious from the formula which is the only hard part of 4.).

$$F_{n+1}(t) = \frac{1}{(n+1)\sin\left(\frac{t}{2}\right) 2\pi} \sum_{k=0}^{n} \sin\left(\left(k+\frac{1}{2}\right)t\right)$$

$$= \frac{1}{(n+1)\sin^2\left(\frac{t}{2}\right) 2\pi} \sum_{k=0}^{n} \sin\left(\left(k+\frac{1}{2}\right)t\right)\sin\left(\frac{t}{2}\right)$$

Using the identity $\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$ with $a = \left(k+\frac{1}{2}\right)t$ and $b = \frac{t}{2}$, it follows

$$F_{n+1}(t) = \frac{1}{(n+1)\sin^2\left(\frac{t}{2}\right) 4\pi} \sum_{k=0}^{n} \left(\cos(kt) - \cos(k+1)t\right)$$

$$= \frac{1-\cos((n+1)t)}{(n+1)\sin^2\left(\frac{t}{2}\right) 4\pi}$$

which completes the demonstration of 4.).

Next consider 5.). Since F_{n+1} is even it suffices to show

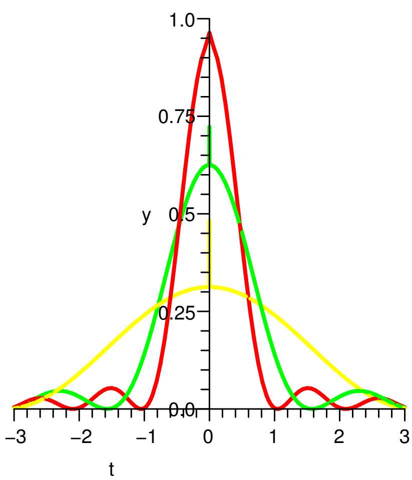
$$\lim_{n\to\infty} \sup \{F_{n+1}(t) : \pi \geq t \geq \delta\} = 0$$

For the given t,

$$F_{n+1}(t) \leq \frac{1-\cos((n+1)t)}{(n+1)\sin^2\left(\frac{\delta}{2}\right) 4\pi} \leq \frac{2}{(n+1)\sin^2\left(\frac{\delta}{2}\right) 4\pi}$$

which shows 5.). This proves the lemma.

Here is a picture of the Fejer kernels for n=2,4,6.



Note how these kernels are nonnegative, unlike the Dirichlet kernels. Also there is a large bump in the center which gets increasingly large as n gets larger. The fact these kernels are nonnegative is what is responsible for the superior ability of the Fejer means to approximate a continuous function.

Theorem 10.6.4 *Let f be a continuous and 2π periodic function. Then*

$$\lim_{n\to\infty} ||f - \sigma_{n+1} f||_0 = 0.$$

Proof: Let $\varepsilon > 0$ be given. Then by part 2. of Lemma 10.6.3,

$$|f(x) - \sigma_{n+1} f(x)| = \left| \int_{-\pi}^{\pi} f(x) F_{n+1}(y)\, dy - \int_{-\pi}^{\pi} F_{n+1}(y) f(x-y)\, dy \right|$$

$$= \left| \int_{-\pi}^{\pi} (f(x) - f(x-y)) F_{n+1}(y)\, dy \right|$$

$$\leq \int_{-\pi}^{\pi} |f(x) - f(x-y)| F_{n+1}(y)\, dy$$

$$= \int_{-\delta}^{\delta} |f(x) - f(x-y)| F_{n+1}(y)\, dy + \int_{\delta}^{\pi} |f(x) - f(x-y)| F_{n+1}(y)\, dy$$
$$+ \int_{-\pi}^{-\delta} |f(x) - f(x-y)| F_{n+1}(y)\, dy$$

Since F_{n+1} is even and $|f|$ is continuous and periodic, hence bounded by some constant M the above is dominated by

$$\leq \int_{-\delta}^{\delta} |f(x) - f(x-y)| F_{n+1}(y)\, dy + 4M \int_{\delta}^{\pi} F_{n+1}(y)\, dy$$

Now choose δ such that for all x, it follows that if $|y| < \delta$ then

$$|f(x) - f(x-y)| < \varepsilon/2.$$

This can be done because f is uniformly continuous on $[-\pi, \pi]$ by Theorem 6.6.2 on Page 102. Since it is periodic, it must also be uniformly continuous on $\mathbb{R}$. (why?) Therefore, for this δ, this has shown that for all x

$$|f(x) - \sigma_{n+1} f(x)| \leq \varepsilon/2 + 4M \int_{\delta}^{\pi} F_{n+1}(y)\, dy$$

and now by Lemma 10.6.3 it follows

$$||f - \sigma_{n+1} f||_0 \leq \varepsilon/2 + \frac{8M\pi}{(n+1)\sin^2\left(\frac{\delta}{2}\right) 4\pi} < \varepsilon$$

provided n is large enough. This proves the theorem.

10.6.2 Mean Square Approximation

The partial sums of the Fourier series of f do a better job approximating f in the mean square sense than any other linear combination of the functions, $e^{ik\theta}$ for $|k| \leq n$. This will be shown next. It is nothing but a simple computation. Recall the Fourier coefficients are

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}$$

Then using this fact as needed, consider the following computation.

$$\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{k=-n}^{n} b_k e^{ik\theta} \right|^2 d\theta$$

$$= \int_{-\pi}^{\pi} \left(f(\theta) - \sum_{k=-n}^{n} b_k e^{ik\theta} \right) \left(\overline{f(\theta)} - \sum_{l=-n}^{n} \overline{a_l} e^{-il\theta} \right) d\theta$$

$$= \int_{-\pi}^{\pi} \left(|f(\theta)|^2 + \left(\sum_{k=-n}^{n} b_k e^{ik\theta} \right) \left(\sum_{l=-n}^{n} \overline{b_l} e^{-il\theta} \right) \right.$$
$$\left. - f(\theta) \sum_{l=-n}^{n} \overline{b_l} e^{-il\theta} - \overline{f(\theta)} \sum_{k=-n}^{n} b_k e^{ik\theta} \right) d\theta$$

$$= \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta + \sum_{kl} b_k \overline{b_l} \int_{-\pi}^{\pi} e^{ik\theta} e^{-il\theta} d\theta - 2\pi \sum_l \overline{b_l} a_l - 2\pi \sum_k b_k \overline{a_k}$$

Then adding and subtracting $2\pi \sum_k |a_k|^2$,

$$= \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - 2\pi \sum_k |a_k|^2 + 2\pi \sum_k |b_k|^2$$
$$-2\pi \sum_l \overline{b_l} a_l - 2\pi \sum_k b_k \overline{a_k} + 2\pi \sum_k |a_k|^2$$

$$= \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - 2\pi \sum_k |a_k|^2 + 2\pi \left(\sum_k (b_k - a_k)\left(\overline{b_k} - \overline{a_k}\right) \right)$$

$$= \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - 2\pi \sum_k |a_k|^2 + 2\pi \sum_k |b_k - a_k|^2$$

Therefore, to make

$$\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{k=-n}^{n} b_k e^{ik\theta} \right|^2 d\theta$$

as small as possible for all choices of b_k, one should let $b_k = a_k$, the k^{th} Fourier coefficient. Stated another way,

$$\int_{-\pi}^{\pi} \left| f(\theta) - \sum_{k=-n}^{n} b_k e^{ik\theta} \right|^2 d\theta \geq \int_{-\pi}^{\pi} |f(\theta) - S_n f(\theta)|^2 d\theta$$

for any choice of b_k. In particular,

$$\int_{-\pi}^{\pi} |f(\theta) - \sigma_{n+1} f(\theta)|^2 \, d\theta \geq \int_{-\pi}^{\pi} |f(\theta) - S_n f(\theta)|^2 \, d\theta. \tag{10.23}$$

Also,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta) \overline{S_n f(\theta)} d\theta &= \int_{-\pi}^{\pi} \sum_{k=-n}^{n} \overline{a_k} f(\theta) e^{-ik\theta} d\theta \\
&= \sum_{k=-n}^{n} \overline{a_k} \overbrace{\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta}^{=2\pi a_k} \\
&= 2\pi \sum_{k=-n}^{n} |a_k|^2
\end{aligned}$$

Similarly,

$$\int_{-\pi}^{\pi} \overline{f(\theta)} S_n f(\theta) \, d\theta = 2\pi \sum_{k=-n}^{n} |a_k|^2$$

and a simple computation of the above sort shows that also

$$\int_{-\pi}^{\pi} S_n f(\theta) \overline{S_n f(\theta)} d\theta = 2\pi \sum_{k=-n}^{n} |a_k|^2 .$$

Therefore,

$$0 \leq \int_{-\pi}^{\pi} (f(\theta) - S_n f(\theta)) \left(\overline{f(\theta)} - \overline{S_n f(\theta)} \right) d\theta$$

$$\begin{aligned}
&= \int_{-\pi}^{\pi} |f(\theta)|^2 + |S_n f(\theta)|^2 - \overline{f(\theta)} S_n f(\theta) - f(\theta) \overline{S_n f(\theta)} d\theta \\
&= \int_{-\pi}^{\pi} |f(\theta)|^2 - |S_n f(\theta)|^2 \, d\theta
\end{aligned}$$

showing

$$2\pi \sum_{k=-n}^{n} |a_k|^2 \leq \int_{-\pi}^{\pi} |S_n f(\theta)|^2 \, d\theta \leq \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta \tag{10.24}$$

Now it is easy to prove the following fundamental theorem.

Theorem 10.6.5 *Let $f \in R([-\pi, \pi])$ and it is periodic of period 2π. Then*

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |f - S_n f|^2 \, dx = 0.$$

Proof: First assume f is continuous and 2π periodic. Then by 10.23

$$\begin{aligned}
\int_{-\pi}^{\pi} |f - S_n f|^2 \, dx &\leq \int_{-\pi}^{\pi} |f - \sigma_{n+1} f|^2 \, dx \\
&\leq \int_{-\pi}^{\pi} ||f - \sigma_{n+1} f||_0^2 \, dx = 2\pi \, ||f - \sigma_{n+1} f||_0^2
\end{aligned}$$

and the last expression converges to 0 by Theorem 10.6.4.

Next suppose $f \in R([-\pi,\pi])$ and $|f(x)| < M$ for all x. Then the construction used in Lemma 10.3.2 yields a continuous function, h which vanishes off some closed interval contained in $(-\pi,\pi)$ such that

$$\frac{\varepsilon}{32M^2} > \int_{-\pi}^{\pi} |f(x) - h(x)|\, dx \geq \frac{1}{2M}\int_{-\pi}^{\pi} |f(x) - h(x)|^2\, dx. \tag{10.25}$$

Since h vanishes off some closed interval contained in $(-\pi,\pi)$, if h is extended off $[-\pi,\pi]$ to be 2π periodic, it follows the resulting function, still denoted by h, is continuous. Then using the inequality (For a better inequality, see Problem 2.)

$$(a+b+c)^2 \leq 4\left(a^2+b^2+c^2\right)$$

$$\int_{-\pi}^{\pi} |f - S_n f|^2\, dx = \int_{-\pi}^{\pi} (|f-h| + |h - S_n h| + |S_n h - S_n f|)^2\, dx$$

$$\leq 4\int_{-\pi}^{\pi} \left(|f-h|^2 + |h - S_n h|^2 + |S_n h - S_n f|^2\right) dx$$

and from 10.24 and 10.25 this is no larger than

$$\begin{aligned} & 8\int_{-\pi}^{\pi} |f-h|^2\, dx + 4\int_{-\pi}^{\pi} |h - S_n h|^2\, dx \\ < \; & 16M\frac{\varepsilon}{16M} + 4\int_{-\pi}^{\pi} |h - S_n h|^2\, dx \\ = \; & \varepsilon + 4\int_{-\pi}^{\pi} |h - S_n h|^2\, dx \end{aligned}$$

and by the first part, this last term converges to 0 as $n \to \infty$. Therefore, since ε is arbitrary, this shows that for n large enough,

$$\int_{-\pi}^{\pi} |f - S_n f|^2\, dx < \varepsilon$$

This proves the theorem.

10.7 Exercises

1. Suppose f has infinitely many derivatives and is also periodic with period 2π. Let the Fourier series of f be

$$\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$

Show that

$$\lim_{k\to\infty} k^m a_k = \lim_{k\to\infty} k^m a_{-k} = 0$$

for every $m \in \mathbb{N}$.

2. The proof of Theorem 10.6.5 used the inequality $(a+b+c)^2 \leq 4\left(a^2+b^2+c^2\right)$ whenever a, b and c are nonnegative numbers. In fact the 4 can be replaced with 3. Show this is true.

3. Let f be a continuous function defined on $[-\pi, \pi]$. Show there exists a polynomial, p such that $||p - f|| < \varepsilon$ where
$$||g|| \equiv \sup \{|g(x)| : x \in [-\pi, \pi]\}.$$
Extend this result to an arbitrary interval. This is another approach to the Weierstrass approximation theorem. **Hint:** First find a linear function, $ax + b = y$ such that $f - y$ has the property that it has the same value at both ends of $[-\pi, \pi]$. Therefore, you may consider this as the restriction to $[-\pi, \pi]$ of a continuous periodic function, F. Now find a trig polynomial,
$$\sigma(x) \equiv a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$
such that $||\sigma - F|| < \frac{\varepsilon}{3}$. Recall 10.4. Now consider the power series of the trig functions making use of the error estimate for the remainder after m terms.

4. The inequality established above,
$$2\pi \sum_{k=-n}^{n} |a_k|^2 \leq \int_{-\pi}^{\pi} |S_n f(\theta)|^2 \, d\theta \leq \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$$
is called Bessel's inequality. Use this inequality to give an easy proof that for all $f \in R([-\pi, \pi])$,
$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) e^{inx} dx = 0.$$
Recall that in the Riemann Lebesgue lemma $|f| \in R((a, b])$ so while this exercise is easier, it lacks the generality of the earlier proof.

5. Suppose $f(x) = \mathcal{X}_{[a,b]}(x)$. Show
$$\lim_{\alpha \to \infty} \int_a^b f(x) \sin(\alpha x + \beta) \, dx = 0.$$
Use this to construct a much simpler proof of the Riemann Lebesgue lemma than that given in the chapter. **Hint:** Show it works for f a step function and then obtain the conclusion for $|f| \in R((a, b])$.

6. Let $f(x) = x$ for $x \in (-\pi, \pi)$ and extend to make the resulting function defined on $\mathbb{R}$ and periodic of period 2π. Find the Fourier series of f. Verify the Fourier series converges to the midpoint of the jump and use this series to find a nice formula for $\frac{\pi}{4}$. **Hint:** For the last part consider $x = \frac{\pi}{2}$.

7. Let $f(x) = x^2$ on $(-\pi, \pi)$ and extend to form a 2π periodic function defined on $\mathbb{R}$. Find the Fourier series of f. Now obtain a famous formula for $\frac{\pi^2}{6}$ by letting $x = \pi$.

8. Let $f(x) = \cos x$ for $x \in (0, \pi)$ and define $f(x) \equiv -\cos x$ for $x \in (-\pi, 0)$. Now extend this function to make it 2π periodic. Find the Fourier series of f.

9. Suppose $f, g \in R([-\pi, \pi])$. Show
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f\overline{g} dx = \sum_{k=-\infty}^{\infty} \alpha_k \overline{\beta_k},$$
where α_k are the Fourier coefficients of f and β_k are the Fourier coefficients of g.

10. Recall the partial summation formula, called the Dirichlet formula which says that

$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k \left(b_k - b_{k+1}\right).$$

Here $A_q \equiv \sum_{k=1}^{q} a_k$. Also recall Dirichlet's test which says that if $\lim_{k\to\infty} b_k = 0$, A_k are bounded, and $\sum |b_k - b_{k+1}|$ converges, then $\sum a_k b_k$ converges. Show the partial sums of $\sum_k \sin kx$ are bounded for each $x \in \mathbb{R}$. Using this fact and the Dirichlet test above, obtain some theorems which will state that $\sum_k a_k \sin kx$ converges for all x.

11. Let $\{a_n\}$ be a sequence of positive numbers having the property that $\lim_{n\to\infty} na_n = 0$ and for all $n \in \mathbb{N}$, $na_n \geq (n+1) a_{n+1}$. Show that if this is so, it follows that the series, $\sum_{k=1}^{\infty} a_n \sin nx$ converges uniformly on $\mathbb{R}$. This is a variation of a very interesting problem found in Apostol's book, [3]. **Hint:** Use the Dirichlet formula of Problem 10 on $\sum ka_k \frac{\sin kx}{k}$ and show the partial sums of $\sum \frac{\sin kx}{k}$ are bounded independent of x. To do this, you might argue the maximum value of the partial sums of this series occur when $\sum_{k=1}^{n} \cos kx = 0$. Sum this series by considering the real part of the geometric series, $\sum_{k=1}^{q} \left(e^{ix}\right)^k$ and then show the partial sums of $\sum \frac{\sin kx}{k}$ are Riemann sums for a certain finite integral.

12. The problem in Apostol's book mentioned in Problem 11 does not require na_n to be decreasing and is as follows. Let $\{a_k\}_{k=1}^{\infty}$ be a decreasing sequence of nonnegative numbers which satisfies $\lim_{n\to\infty} na_n = 0$. Then

$$\sum_{k=1}^{\infty} a_k \sin(kx)$$

converges uniformly on $\mathbb{R}$. You can find this problem worked out completely in Jones [21]. Fill in the details to the following argument or something like it to obtain a proof. First show that for $p < q$, and $x \in (0, \pi)$,

$$\left| \sum_{k=p}^{q} a_k \sin(kx) \right| \leq 3a_p \csc\left(\frac{x}{2}\right). \tag{10.26}$$

To do this, use summation by parts using the formula

$$\sum_{k=p}^{q} \sin(kx) = \frac{\cos\left(\left(p - \frac{1}{2}\right)x\right) - \cos\left(\left(q + \frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)},$$

which you can establish by taking the imaginary part of a geometric series of the form $\sum_{k=1}^{q} \left(e^{ix}\right)^k$ or else the approach used above to find a formula for the Dirichlet kernel. Now define

$$b(p) \equiv \sup\{na_n : n \geq p\}.$$

Thus $b(p) \to 0$, $b(p)$ is decreasing in p, and if $k \geq n$, $a_k \leq b(n)/k$. Then from 10.26

and the assumption $\{a_k\}$ is decreasing,

$$\begin{aligned}\left|\sum_{k=p}^{q} a_k \sin(kx)\right| &\leq \left|\sum_{k=p}^{m} a_k \sin(kx)\right| + \overbrace{\left|\sum_{k=m+1}^{q} a_k \sin(kx)\right|}^{\equiv 0 \text{ if } m=q} \\ &\leq \begin{cases} \sum_{k=p}^{m} \frac{b(k)}{k} \left|\sin(kx)\right| + 3a_p \csc\left(\frac{x}{2}\right) \text{ if } m < q \\ \sum_{k=p}^{q} \frac{b(k)}{k} \left|\sin(kx)\right| \text{ if } m = q. \end{cases} \\ &\leq \begin{cases} \sum_{k=p}^{m} \frac{b(k)}{k} kx + 3a_p \frac{2\pi}{x} \text{ if } m < q \\ \sum_{k=p}^{q} \frac{b(k)}{k} kx \text{ if } m = q \end{cases} \qquad (10.27)\end{aligned}$$

where this uses the inequalities

$$\sin\left(\frac{x}{2}\right) \geq \frac{x}{2\pi}, \left|\sin(x)\right| \leq |x| \text{ for } x \in (0, \pi).$$

There are two cases to consider depending on whether $x \leq 1/q$. First suppose that $x \leq 1/q$. Then let $m = q$ and use the bottom line of 10.27 to write that in this case,

$$\left|\sum_{k=p}^{q} a_k \sin(kx)\right| \leq \frac{1}{q} \sum_{k=p}^{q} b(k) \leq b(p).$$

If $x > 1/q$, then $q > 1/x$ and you use the top line of 10.27 picking m such that

$$q > \frac{1}{x} \geq m \geq \frac{1}{x} - 1.$$

Then in this case,

$$\begin{aligned}\left|\sum_{k=p}^{q} a_k \sin(kx)\right| &\leq \sum_{k=p}^{m} \frac{b(k)}{k} kx + 3a_p \frac{2\pi}{x} \\ &\leq b(p)\, x\,(m-p) + 6\pi a_p (m+1) \\ &\leq b(p)\, x \left(\frac{1}{x}\right) + 6\pi \frac{b(p)}{m+1} (m+1) \leq 25 b(p).\end{aligned}$$

Therefore, the partial sums of the series, $\sum a_k \sin kx$ form a uniformly Cauchy sequence and must converge uniformly on $(0, \pi)$. Now explain why this implies the series converges uniformly on $\mathbb{R}$.

13. Suppose $f(x) = \sum_{k=1}^{\infty} a_k \sin kx$ and that the convergence is uniform. Recall something like this holds for power series. Is it reasonable to suppose that $f'(x) = \sum_{k=1}^{\infty} a_k k \cos kx$? Explain.

14. Suppose $|u_k(x)| \leq K_k$ for all $x \in D$ where

$$\sum_{k=-\infty}^{\infty} K_k = \lim_{n \to \infty} \sum_{k=-n}^{n} K_k < \infty.$$

Show that $\sum_{k=-\infty}^{\infty} u_k(x)$ converges converges uniformly on D in the sense that for all $\varepsilon > 0$, there exists N such that whenever $n > N$,

$$\left|\sum_{k=-\infty}^{\infty} u_k(x) - \sum_{k=-n}^{n} u_k(x)\right| < \varepsilon$$

for all $x \in D$. This is called the Weierstrass M test.

15. Let $a_k, b_k \geq 0$. Show the Cauchy Schwarz inequality

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}$$

To do this note that

$$p(t) \equiv \sum_{k=1}^{n} (a_k + t b_k)^2 \geq 0$$

for all t. Now pick an auspicious value of t, perhaps the value at which $p(t)$ achieves its minimum.

16. Suppose f is a differentiable function of period 2π and suppose that both f and f' are in $R([-\pi, \pi])$ such that for all $x \in (-\pi, \pi)$ and y sufficiently small,

$$f(x+y) - f(x) = \int_x^{x+y} f'(t)\, dt.$$

Show that the Fourier series of f converges uniformly to f. **Hint:** First show using the Dini criterion that $S_n f(x) \to f(x)$ for all x. Next let $\sum_{k=-\infty}^{\infty} a_k e^{ikx}$ be the Fourier series for f. Then from the definition of a_k, show that for $k \neq 0, a_k = \frac{1}{ik} a'_k$ where a'_k is the Fourier coefficient of f'. Now use the Bessel's inequality to argue that $\sum_{k=-\infty}^{\infty} |a'_k|^2 < \infty$ and then show this implies $\sum |a_k| < \infty$. You might want to use the Cauchy Schwarz inequality in Problem 15 to do this part. Then using the version of the Weierstrass M test given in Problem 14 obtain uniform convergence of the Fourier series to f.

17. Let f be a function defined on $\mathbb{R}$. Then f is even if $f(\theta) = f(-\theta)$ for all $\theta \in \mathbb{R}$. Also f is called odd if for all $\theta \in \mathbb{R}$, $-f(\theta) = f(-\theta)$. Now using the Weierstrass approximation theorem show directly that if h is a continuous even 2π periodic function, then for every $\varepsilon > 0$ there exists an m and constants, $a_0, \cdots, a_m$ such that

$$\left| h(\theta) - \sum_{k=0}^{m} a_k \cos^k(\theta) \right| < \varepsilon$$

for all $\theta \in \mathbb{R}$. **Hint:** Note the function arccos is continuous and maps $[-1, 1]$ onto $[0, \pi]$. Using this show you can define g a continuous function on $[-1, 1]$ by $g(\cos\theta) = h(\theta)$ for θ on $[0, \pi]$. Now use the Weierstrass approximation theorem on $[-1, 1]$.

18. Show that if f is any odd 2π periodic function, then its Fourier series can be simplified to an expression of the form

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

and also $f(m\pi) = 0$ for all $m \in \mathbb{N}$.

19. Consider the symbol $\sum_{k=1}^{\infty} a_n$. The infinite sum might not converge. Summability methods are systematic ways of assigning a number to such a symbol. The n^{th} Ceasaro mean σ_n is defined as the average of the first n partial sums of the series. Thus

$$\sigma_n \equiv \frac{1}{n} \sum_{k=1}^{n} S_k$$

where

$$S_k \equiv \sum_{j=1}^{k} a_j.$$

Show that if $\sum_{k=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} \sigma_n$ also exists and equals the same thing. Next find an example where, although $\sum_{k=1}^{\infty} a_n$ fails to converge, $\lim_{n\to\infty} \sigma_n$ does exist. This summability method is called Ceasaro summability. Recall the Fejer means were obtained in just this way.

20. Let $0 < r < 1$ and for f a periodic function of period 2π where $f \in R([-\pi,\pi])$, consider

$$A_r f(\theta) \equiv \sum_{k=-\infty}^{\infty} r^{|k|} a_k e^{ik\theta}$$

where the a_k are the Fourier coefficients of f. Show that if f is continuous, then

$$\lim_{r\to 1-} A_r f(\theta) = f(\theta).$$

Hint: You need to find a kernel and write as the integral of the kernel convolved with f. Then consider properties of this kernel as was done with the Fejer kernel. In carrying out the details, you need to verify the convergence of the series is uniform in some sense in order to switch the sum with an integral.

21. Recall the Dirichlet kernel is

$$D_n(t) \equiv (2\pi)^{-1} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}$$

and it has the property that $\int_{-\pi}^{\pi} D_n(t)\,dt = 1$. Show first that this implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(nt)\cos(t/2)}{\sin(t/2)}dt = 1$$

and this implies

$$\frac{1}{\pi}\int_{0}^{\pi} \frac{\sin(nt)\cos(t/2)}{\sin(t/2)} = 1.$$

Next change the variable to show the integral equals

$$\frac{1}{\pi}\int_{0}^{n\pi} \frac{\sin(u)\cos(u/2n)}{\sin(u/2n)}\frac{1}{n}du$$

Now show that

$$\lim_{n\to\infty} \frac{\sin(u)\cos(u/2n)}{\sin(u/2n)}\frac{1}{n} = \frac{2\sin u}{u}$$

Next show that

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{0}^{n\pi} \frac{2\sin u}{u}du = \lim_{n\to\infty} \frac{1}{\pi}\int_{0}^{n\pi} \frac{\sin(u)\cos(u/2n)}{\sin(u/2n)}\frac{1}{n}du = 1$$

Finally show

$$\lim_{R\to\infty}\int_{0}^{R} \frac{\sin u}{u}du = \frac{\pi}{2} \equiv \int_{0}^{\infty} \frac{\sin u}{u}du$$

This is a very important improper integral.

22. To work this problem, you should first review Problem 25 on Page 212 about interchanging the order of iterated integrals. Suppose f is Riemann integrable on every finite interval, bounded, and

$$\lim_{R\to\infty}\int_0^R |f(t)|\,dt<\infty,\ \lim_{R\to\infty}\int_{-R}^0 |f(t)|\,dt<\infty.$$

Show that

$$\lim_{R\to\infty}\int_0^R f(t)\,e^{ist}dt,\ \lim_{R\to\infty}\int_{-R}^0 f(t)\,e^{-its}dt$$

both exist. Define

$$\begin{aligned}\lim_{R\to\infty}\int_0^R f(t)\,e^{ist}dt &\equiv \int_0^\infty f(t)\,e^{-its}dt\\ \lim_{R\to\infty}\int_{-R}^0 f(t)\,e^{-its}dt &\equiv \int_{-\infty}^0 f(t)\,e^{-its}dt\end{aligned}$$

and

$$\int_{-\infty}^\infty f(t)\,e^{-ist}dt\equiv\int_0^\infty f(t)\,e^{-its}dt+\int_{-\infty}^0 f(t)\,e^{-its}dt$$

Now suppose the Dini condition on f, that

$$t\to\frac{f(x-t)+f(x+t)-(f(x-)+f(x+))}{2t}$$

is a function in $R((0,1])$. This happens, for example if for $t>0$ and small,

$$|f(x+t)-f(x+)|\le Ct^\theta \text{ for } \theta>0$$

and if

$$|f(x-t)-f(x-)|\le Ct^\alpha \text{ for } \alpha>0$$

Now show using Problem 25 on Page 212 (You fill in the details including modifying things so that the result of this problem applies.) to conclude

$$\frac{1}{2\pi}\int_{-R}^R\int_{-\infty}^\infty f(t)\,e^{-its}dte^{isx}ds=\frac{1}{2\pi}\int_{-R}^R\int_{-\infty}^\infty f(x-u)\,e^{isu}duds$$

$$\begin{aligned}&=\frac{1}{2\pi}\int_{-\infty}^\infty f(x-u)\int_{-R}^R e^{isu}dsdu=\frac{1}{\pi}\int_{-\infty}^\infty f(x-u)\frac{\sin Ru}{u}du\\ &=\frac{2}{\pi}\int_0^\infty\frac{\sin u}{u}\frac{f\left(x-\frac{u}{R}\right)+f\left(x+\frac{u}{R}\right)}{2}du\end{aligned}$$

Conclude using Problem 21

$$\frac{1}{2\pi}\int_{-R}^R\int_{-\infty}^\infty f(t)\,e^{-its}dte^{isx}ds-\frac{f(x-)+f(x+)}{2}\tag{10.28}$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin u}{u}\left(\frac{f\left(x-\frac{u}{R}\right)+f\left(x+\frac{u}{R}\right)}{2}-\frac{f(x-)+f(x+)}{2}\right)du$$

Explain how the above equals an expression of the form

$$\frac{2}{\pi}\int_0^R \frac{\sin u}{u}\left(\frac{f\left(x-\frac{u}{R}\right)+f\left(x+\frac{u}{R}\right)}{2}-\frac{f\left(x-\right)+f\left(x+\right)}{2}\right)du+e\left(R\right)$$

where $e(R)\to 0$ as $R\to\infty$. Now change the variable letting $u/R=t$ and conclude 10.28 equals

$$\frac{2}{\pi}\int_0^1 \sin\left(Rt\right)\left(\frac{f\left(x-t\right)+f\left(x+t\right)-\left(f\left(x-\right)+f\left(x+\right)\right)}{2t}\right)dt+e\left(R\right)$$

Now apply the Riemann Lebesgue lemma to conclude

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^R\int_{-\infty}^{\infty} f\left(t\right)e^{-its}dte^{isx}ds=\frac{f\left(x-\right)+f\left(x+\right)}{2}.$$

The Fourier transform is defined as

$$F\left(s\right)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\left(t\right)e^{-its}dt$$

and this has shown that under the Dini condition described above and for the sort of f defined above,

$$\frac{f\left(x-\right)+f\left(x+\right)}{2}=\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^R F\left(s\right)e^{isx}ds$$

This is the Fourier inversion formula.

The Generalized Riemann Integral

11.1 Definitions And Basic Properties

This chapter is on the generalized Riemann integral. The Riemann Darboux integral presended earlier has been obsolete for over 100 years. The integral of this chapter is certainly not obsolete and is in certain important ways the very best integral currently known. This integral is called the generalized Riemann integral, also the Henstock Kurzweil integral after the two people who invented it and sometimes the gauge integral. Other books which discuss this integral are the books by Bartle [5], Bartle and Sherbert, [6], Henstock [19], or McLeod [26]. Considerably more is presented in these references. In what follows, F will be an increasing function, the most important example being $F(x) = x$.

Definition 11.1.1 *Let $[a,b]$ be a closed and bounded interval. A tagged division*[1] *of $[a,b] = I$ is a set of the form $P \equiv \{(I_i, t_i)\}_{i=1}^n$ where $t_i \in I_i = [x_{i-1}, x_i]$, and $a = x_{i-1} < \cdots < x_n = b$. Let the t_i be referred to as the tags. A function, $\delta : I \to (0,\infty)$ is called a gauge function or simply gauge for short. A tagged division, P is called δ fine if*

$$I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)] .$$

A δ fine division, is understood to be tagged. More generally, a collection, $\{(I_i, t_i)\}_{i=1}^p$ is δ fine if the above inclusion holds for each of these intervals and their interiors are disjoint even if their union is not equal to the whole interval, $[a,b]$.

The following fundamental result is essential.

Proposition 11.1.2 *Let $[a,b]$ be an interval and let δ be a gauge function on $[a,b]$. Then there exists a δ fine tagged division of $[a,b]$.*

Proof: Suppose not. Then one of $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$ must fail to have a δ fine tagged division because if they both had such a δ fine division, the union of the two δ fine divisions would be a δ fine division of $[a,b]$. Denote by I_1 the interval which does not have a δ fine division. Then repeat the above argument, dividing I_1 into two equal intervals and pick the one, I_2 which fails to have a δ fine division. Continue this way to get a nested sequence of closed intervals, $\{I_i\}$ having the property that each interval in the set fails to have a δ fine division and $\text{diam}(I_i) \to 0$. Therefore,

$$\cap_{i=1}^{\infty} I_i = \{x\}$$

[1] In beginning calculus books, this is often called a partition. The word, division is a much better word to use. what the x_i do is to "divide" the interval into little subintervals.

where $x \in [a,b]$. Now $[x - \delta(x), x + \delta(x)]$ must contain some I_k because the diameters of these intervals converge to zero. It follows that $\{(I_k, x)\}$ is a δ fine division of I_k, contrary to the construction which required that none of these intervals had a δ fine division. This proves the proposition.

With this proposition and definition, it is time to define the generalized Riemann integral. The functions being integrated typically have values in $\mathbb{R}$ or $\mathbb{C}$ but there is no reason to restrict to this situation and so in the following definition, X will denote the space in which f has its values. For example, X could be $\mathbb{R}^p$ which becomes important in multivariable calculus. For now, just think $\mathbb{R}$ if you like. It will be assumed Cauchy sequences converge and there is something like the absolute value, called a norm although it is possible to generalize even further.

Definition 11.1.3 *Let X be a complete normed vector space. (For example, $X = \mathbb{R}$ or $X = \mathbb{C}$ or $X = \mathbb{R}^p$.) Then $f : [a,b] \to X$ is generalized Riemann integrable, written as $f \in R^* [a,b]$ if there exists $R \in X$ such that for all $\varepsilon > 0$, there exists a gauge, δ, such that if $P \equiv \{(I_i, t_i)\}_{i=1}^n$ is δ fine then defining, $S(P, f)$ by*

$$S(P,f) \equiv \sum_{i=1}^{n} f(t_i)\,\Delta F_i,$$

where if $I_i = [x_{i-1}, x_i]$,

$$\Delta F_i \equiv F(x_i) - F(x_{i-1})$$

it follows

$$|S(P,f) - R| < \varepsilon.$$

Then

$$\int_I f dF \equiv \int_a^b f dF \equiv R.$$

Here $|\cdot|$ refers to the norm on X for $\mathbb{R}$ this is just the absolute value.

Note that if P is δ_1 fine and $\delta_1 \leq \delta$ then it follows P is also δ fine.

How does this relate to the usual Riemann integral discussed above in Theorem 9.4.2 and the definition of the Riemann integral given before this?

To begin with, is there a way to tell whether a given function is in $R^*[a,b]$? The following Cauchy criterion is useful to make this determination.

Proposition 11.1.4 *A function $f : [a,b] \to X$ is in $R^*[a,b]$ if and only if for every $\varepsilon > 0$, there exists a gauge function, δ_ε such that if P and Q are any two divisions which are δ_ε fine, then*

$$|S(P,f) - S(Q,f)| < \varepsilon.$$

Proof: Suppose first that $f \in R^*[a,b]$. Then there exists a gauge, δ_ε, and an element of X, R, such that if P is δ_ε fine, then

$$|R - S(P,f)| < \varepsilon/2.$$

Now let P, Q be two such δ_ε fine divisions. Then

$$\begin{aligned} |S(P,f) - S(Q,f)| &\leq |S(P,f) - R| + |R - S(Q,f)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, suppose the condition of the proposition holds. Let $\varepsilon_n \to 0+$ as $n \to \infty$ and let δ_{ε_n} denote the gauge which goes with ε_n. Without loss of generality, assume that δ_{ε_n}

is decreasing in n. Let R_{ε_n} denote the closure of all the sums, $S(P,f)$ where P is δ_{ε_n} fine. From the condition, it follows $\text{diam}(R_{\varepsilon_n}) \leq \varepsilon_n$ and that these closed sets are nested in the sense that $R_{\varepsilon_n} \supseteq R_{\varepsilon_{n+1}}$ because δ_{ε_n} is decreasing in n. Therefore, there exists a unique, $R \in \cap_{n=1}^{\infty} R_{\varepsilon_n}$. To see this, let $r_n \in R_{\varepsilon_n}$. Then since the diameters of the R_{ε_n} are converging to 0, $\{r_n\}$ is a Cauchy sequence which must converge to some $R \in X$. Since R_{ε_n} is closed, it follows $R \in R_{\varepsilon_n}$ for each n. Letting $\varepsilon > 0$ be given, there exists $\varepsilon_n < \varepsilon$ and for P a δ_{ε_n} fine division,

$$|S(P,f) - R| \leq \varepsilon_n < \varepsilon.$$

Therefore, $R = \int_I f$. This proves the proposition.

Are there examples of functions which are in $R^*[a,b]$? Are there examples of functions which are not? It turns out the second question is harder than the first although it is very easy to answer this question in the case of the obsolete Riemann integral. The generalized Riemann integral is a vastly superior integral which can integrate a very impressive collection of functions. Consider the first question. Recall the definition of the Riemann integral given above which is listed here for convenience.

Definition 11.1.5 *A bounded function, f defined on $[a,b]$ is said to be Riemann Stieltjes integrable if there exists a number, I with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$P \equiv \{x_0, x_1, \cdots, x_n\}$$

is any partition having $||P|| < \delta$, and $z_i \in [x_{i-1}, x_i]$,

$$\left| I - \sum_{i=1}^{n} f(z_i)(F(x_i) - F(x_{i-1})) \right| < \varepsilon.$$

The number $\int_a^b f(x)\, dF$ is defined as I.

First note that if $\delta > 0$ and if every interval in a division has length less than δ then the division is δ fine. In fact, you could pick the tags as any point in the intervals. Then the following theorem follows immediately.

Theorem 11.1.6 *Let f be Riemann Stieltjes integrable according to Definition 11.1.5. Then f is generalized Riemann integrable and the integrals are the same.*

Proof: Just let the gauge functions be constant functions.

In particular, the following important theorem follows from Theorem 9.4.6.

Theorem 11.1.7 *Let f be continuous on $[a,b]$ and let F be any increasing integrator. Then $f \in R^*[a,b]$.*

Many functions other than continuous ones are integrable however. In fact, it is fairly difficult to come up with easy examples because this integral can integrate almost anything you can imagine, including the function which equals 1 on the rationals and 0 on the irrationals which is not Riemann integrable. This will be shown later.

The integral is linear. This will be shown next.

Theorem 11.1.8 *Suppose α and β are constants and that f and g are in $R^*[a,b]$. Then $\alpha f + \beta g \in R^*[a,b]$ and*

$$\int_I (\alpha f + \beta g)\, dF = \alpha \int_I f dF + \beta \int_I g dF.$$

Proof: Let $\eta = \frac{\varepsilon}{|\beta|+|\alpha|+1}$ and choose gauges, δ_g and δ_f such that if P is δ_g fine,

$$\left| S(P,g) - \int_I g dF \right| < \eta$$

and that if P is δ_f fine,

$$\left| S(P,f) - \int_I f dF \right| < \eta.$$

Now let $\delta = \min(\delta_g, \delta_f)$. Then if P is δ fine the above inequalities both hold. Therefore, from the definition of $S(P,f)$,

$$S(P, \alpha f + \beta g) = \alpha S(P,f) + \beta S(P,g)$$

and so

$$\left| S(P, \alpha f + \beta g) - \left(\beta \int_I g dF + \alpha \int_I f dF \right) \right| \leq \left| \beta S(P,g) - \beta \int_I g dF \right|$$

$$+ \left| \alpha S(P,f) - \alpha \int_I f dF \right| \leq |\beta| \eta + |\alpha| \eta < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows the number, $\beta \int_I g dF + \alpha \int_I f dF$ qualifies in the definition of the generalized Riemann integral and so $\alpha f + \beta g \in R^* [a,b]$ and

$$\int_I (\alpha f + \beta g) dF = \beta \int_I g dF + \alpha \int_I f dF.$$

The following lemma is also very easy to establish from the definition.

Lemma 11.1.9 *If $f \geq 0$ and $f \in R^*[a,b]$, then $\int_I f dF \geq 0$. Also, if f has complex values and is in $R^*[I]$, then both $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $R^*[I]$.*

Proof: To show the first part, let $\varepsilon > 0$ be given and let δ be a gauge function such that if P is δ fine then

$$\left| S(f,P) - \int_I f dF \right| \leq \varepsilon.$$

Since F is increasing, it is clear that $S(f,P) \geq 0$. Therefore,

$$\int_I f dF \geq S(f,P) - \varepsilon \geq -\varepsilon$$

and since ε is arbitrary, it follows $\int_I f dF \geq 0$ as claimed.

To verify the second part, note that by Proposition 11.1.4 there exists a gauge, δ such that if P, Q are δ fine then

$$|S(f,P) - S(f,Q)| < \varepsilon$$

But

$$\begin{aligned} |S(\operatorname{Re} f, P) - S(\operatorname{Re} f, Q)| &= |\operatorname{Re}(S(f,P)) - \operatorname{Re}(S(f,Q))| \\ &\leq |S(f,P) - S(f,Q)| \end{aligned}$$

and so the conditions of Proposition 11.1.4 are satisfied and you can conclude $\operatorname{Re} f \in R^*[I]$. Similar reasoning applies to $\operatorname{Im} f$. This proves the lemma.

Corollary 11.1.10 *If $|f|, f \in R^*[a,b]$, where f has values in $\mathbb{C}$, then*

$$\left|\int_I f dF\right| \leq \int_I |f| \, dF.$$

Proof: Let $|\alpha| = 1$ and $\alpha \int_I f dF = \left|\int_I f dF\right|$. Then by Theorem 11.1.8 and Lemma 11.1.9,

$$\begin{aligned}\left|\int_I f dF\right| &= \int_I \alpha f dF = \int_I (\operatorname{Re}(\alpha f) + i \operatorname{Im}(\alpha f)) \, dF \\ &= \int_I \operatorname{Re}(\alpha f) \, dF + i \int_I \operatorname{Im}(\alpha f) \, dF = \int_I \operatorname{Re}(\alpha f) \, dF \\ &\leq \int_I |\operatorname{Re}(\alpha f)| \, dF \leq \int_I |f| \, dF\end{aligned}$$

Note the assumption that $|f| \in R^*[a,b]$.

The following lemma is also fundamental.

Lemma 11.1.11 *If $f \in R^*[a,b]$ and $[c,d] \subseteq [a,b]$, then $f \in R^*[c,d]$.*

Proof: Let $\varepsilon > 0$ and choose a gauge δ such that if P is a division of $[a,b]$ which is δ fine, then

$$|S(P,f) - R| < \varepsilon/2.$$

Now pick a δ fine division of $[c,d]$, $\{(I_i,t_i)\}_{i=r}^{l}$ and then let $\{(I_i,t_i)\}_{i=1}^{r-1}$ and $\{(I_i,t_i)\}_{i=l+1}^{n}$ be **fixed** δ fine divisions on $[a,c]$ and $[d,b]$ respectively.

Now let P_1 and Q_1 be δ fine divisions of $[c,d]$ and let P and Q be the respective δ fine divisions if $[a,b]$ just described which are obtained from P_1 and Q_1 by adding in $\{(I_i,t_i)\}_{i=1}^{r-1}$ and $\{(I_i,t_i)\}_{i=l+1}^{n}$. Then

$$\varepsilon > |S(Q,f) - S(P,f)| = |S(Q_1,f) - S(P_1,f)|.$$

By the above Cauchy criterion, Proposition 11.1.4, $f \in R^*[c,d]$ as claimed.

Corollary 11.1.12 *Suppose $c \in [a,b]$ and that $f \in R^*[a,b]$. Then $f \in R^*[a,c]$ and $f \in R^*[c,b]$. Furthermore,*

$$\int_I f dF = \int_a^c f dF + \int_c^b f dF.$$

Here $\int_a^c f dF$ means $\int_{[a,c]} f dF$.

Proof: Let $\varepsilon > 0$. Let δ_1 be a gauge function on $[a,c]$ such that whenever P_1 is a δ_1 fine division of $[a,c]$,

$$\left|\int_a^c f dF - S(P_1,f)\right| < \varepsilon/3.$$

Let δ_2 be a gauge function on $[c,b]$ such that whenever P_2 is a δ_2 fine division of $[c,b]$,

$$\left|\int_c^b f dF - S(P_2,f)\right| < \varepsilon/3.$$

Let δ_3 be a gauge function on $[a,b]$ such that if P is a δ_3 fine division of $[a,b]$,

$$\left|\int_a^b f dF - S(P,f)\right| < \varepsilon/3.$$

Now define a gauge function,

$$\delta(x) \equiv \begin{cases} \min(\delta_1, \delta_3) & \text{on } [a,c] \\ \min(\delta_2, \delta_3) & \text{on } [c,b] \end{cases}$$

Then letting P_1 be a δ fine division on $[a,c]$ and P_2 be a δ fine division on $[c,b]$, it follows that $P = P_1 \cup P_2$ is a δ_3 fine division on $[a,b]$ and all the above inequalities hold. Thus noting that $S(P,f) = S(P_1,f) + S(P_2,f)$,

$$\begin{aligned} & \left| \int_I f dF - \left(\int_a^c f dF + \int_c^b f dF \right) \right| \\ \leq & \left| \int_I f dF - (S(P_1,f) + S(P_2,f)) \right| \\ & + \left| S(P_1,f) + S(P_2,f) - \left(\int_a^c f dF + \int_c^b f dF \right) \right| \end{aligned}$$

$$\begin{aligned} \leq & \left| \int_I f dF - S(P,f) \right| + \left| S(P_1,f) - \int_a^c f dF \right| + \left| S(P_2,f) - \int_c^b f dF \right| \\ < & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Since ε is arbitrary, the conclusion of the corollary follows. This proves the corollary.

The following lemma, sometimes called Henstock's lemma is of great significance.

Lemma 11.1.13 *Suppose that $f \in R^*[a,b]$ and that whenever, Q is a δ fine division of I,*

$$\left| S(Q,f) - \int_I f dF \right| \leq \varepsilon.$$

Then if $P = \{(I_i, t_i)\}_{i=1}^n$ is any δ fine division, and $P' = \left\{ \left(I_{i_j}, t_{i_j} \right) \right\}_{j=1}^r$ is a subset of P, then

$$\left| \sum_{j=1}^r f\left(t_{i_j}\right) \Delta F_i - \sum_{j=1}^r \int_{I_{i_j}} f dF \right| \leq \varepsilon.$$

Proof: Let $(J_k, t) \in P \setminus P'$. From Lemma 11.1.11, $f \in R^*[J_k]$. Therefore, letting $\{(J_{ki}, t_{ki})\}_{i=1}^{l_k}$ be a suitable δ fine division of J_k,

$$\left| \int_{J_k} f dF - \sum_{i=1}^{l_k} f(t_{ki}) \Delta F_i \right| < \eta$$

where $\eta > 0$. Then the collection of these δ fine divisions, $\{(J_{ki}, t_{ki})\}_{i=1}^{l_k}$ taken together with $\left\{ \left(I_{i_j}, t_{i_j} \right) \right\}_{j=1}^r$ forms a δ fine division of I denoted by $\widetilde{P}$. Therefore,

$$\varepsilon \geq \left| S\left(\widetilde{P}, f\right) - \int_I f dF \right| = \left| \sum_{j=1}^r f\left(t_{i_j}\right) \Delta F_i - \sum_{j=1}^r \int_{I_{i_j}} f dF \right.$$

$$\left. + \sum_{k \notin \{i_1, \cdots, i_r\}} \sum_{i=1}^{l_k} f(t_{ki}) \Delta F_i - \sum_{k \notin \{i_1, \cdots, i_r\}} \int_{J_k} f dF \right|$$

$$\geq \left| \sum_{j=1}^{r} f\left(t_{i_j}\right) \Delta F_i - \sum_{j=1}^{r} \int_{I_{i_j}} f dF \right| - |P \setminus P'| \eta$$

where $|P \setminus P'|$ denotes the number of intervals contained in $P \setminus P'$. It follows that

$$\varepsilon + |P \setminus P'| \eta \geq \left| \sum_{j=1}^{r} f\left(t_{i_j}\right) \Delta F_i - \sum_{j=1}^{r} \int_{I_{i_j}} f dF \right|$$

and since η is arbitrary, this proves the lemma.

Consider $\{(I_j, t_j)\}_{j=1}^{p}$ a subset of a division of $[a, b]$. If δ is a gauge and $\{(I_j, t_j)\}_{j=1}^{p}$ is δ fine, this can always be considered as a subset of a δ fine division of the whole interval and so the following corollary is immediate. It is this corollary which seems of most use and may also be referred to as Henstock's lemma.

Corollary 11.1.14 *Suppose that $f \in R^* [a, b]$ and that whenever, Q is a δ fine division of I,*

$$\left| S(Q, f) - \int_I f dF \right| \leq \varepsilon.$$

Then if $\{(I_i, t_i)\}_{i=1}^{p}$ is δ fine, it follows that

$$\left| \sum_{j=1}^{p} f(t_j) \Delta F(I_j) - \sum_{j=1}^{p} \int_{I_j} f dF \right| \leq \varepsilon.$$

Here is another corollary in the special case where f has real values.

Corollary 11.1.15 *Suppose $f \in R^* [a, b]$ has values in $\mathbb{R}$ and that*

$$\left| S(P, f) - \int_I f dF \right| \leq \varepsilon$$

for all P which is δ fine. Then if $P = \{(I_i, t_i)\}_{i=1}^{n}$ is δ fine,

$$\sum_{i=1}^{n} \left| f(t_i) \Delta F_i - \int_{I_i} f dF \right| \leq 2\varepsilon. \tag{11.1}$$

Proof: Let $\mathcal{I} \equiv \left\{ i : f(t_i) \Delta F_i \geq \int_{I_i} f dF \right\}$ and let $\mathcal{I}^C \equiv \{1, \cdots, n\} \setminus \mathcal{I}$. Then by Henstock's lemma

$$\left| \sum_{i \in \mathcal{I}} f(t_i) \Delta F_i - \sum_{i \in \mathcal{I}} \int_{I_i} f dF \right| = \sum_{i \in \mathcal{I}} \left| f(t_i) \Delta F_i - \int_{I_i} f dF \right| \leq \varepsilon$$

and

$$\left| \sum_{i \in \mathcal{I}^C} f(t_i) \Delta F_i - \sum_{i \in \mathcal{I}^C} \int_{I_i} f dF \right| = \sum_{i \in \mathcal{I}^C} \left| f(t_i) \Delta F_i - \int_{I_i} f dF \right| \leq \varepsilon$$

so adding these together yields 11.1.

This generalizes immediately to the following.

Corollary 11.1.16 *Suppose $f \in R^*[a,b]$ has values in $\mathbb{C}$ and that*

$$\left| S(P,f) - \int_I f dF \right| \leq \varepsilon \tag{11.2}$$

for all P which is δ fine. Then if $P = \{(I_i, t_i)\}_{i=1}^n$ is δ fine,

$$\sum_{i=1}^n \left| f(t_i) \Delta F_i - \int_{I_i} f dF \right| \leq 4\varepsilon. \tag{11.3}$$

Proof: It is clear that if 11.2 holds, then

$$\left| S(P, \operatorname{Re} f) - \operatorname{Re} \int_I f dF \right| \leq \varepsilon$$

which shows that $\operatorname{Re} \int_I f dF = \int_I \operatorname{Re} f dF$. Similarly $\operatorname{Im} \int_I f dF = \int_I \operatorname{Im} f dF$. Therefore, using Corollary 11.1.15

$$\sum_{i=1}^n \left| \operatorname{Re} f(t_i) \Delta F_i - \int_{I_i} \operatorname{Re} f dF \right| \leq 2\varepsilon$$

and

$$\sum_{i=1}^n \left| i \operatorname{Im} f(t_i) \Delta F_i - \int_{I_i} i \operatorname{Im} f dF \right| \leq 2\varepsilon.$$

Adding and using the triangle inequality, yields 11.3.

Next is a version of the monotone convergence theorem. The monotone convergence theorem is one which says that under suitable conditions, if $f_n \to f$ pointwise, then $\int_I f_n dF \to \int_I f dF$. These sorts of theorems are not available for the beginning calculus type of integral. This is one of the reasons for considering the generalized integral. The following proof is a special case of one presented in McLeod [26].

Theorem 11.1.17 *Let $f_n(x) \geq 0$ and suppose $f_n \in R^*[a,b]$, $\cdots f_n(x) \leq f_{n+1}(x) \cdots$ and that $f(x) = \lim_{n\to\infty} f_n(x)$ for all x and that $f(x)$ has real values. Also suppose that $\left\{ \int_I f_n \right\}_{n=1}^\infty$ is a bounded sequence. Then $f \in R^*[a,b]$ and*

$$\int_I f dF = \lim_{n\to\infty} \int_I f_n dF \tag{11.4}$$

Proof: The proof will be based on the following lemma. The conclusion of this lemma is sometimes referred to as equiintegrable.

Lemma 11.1.18 *In the situation of Theorem 11.1.17 let $\varepsilon > 0$ be given. Then there exists a gauge, δ, such that if P is δ fine, then for all $n \in \mathbb{N}$,*

$$\left| S(P, f_n) - \int_I f_n dF \right| < \varepsilon. \tag{11.5}$$

Proof of the Lemma: Let $3\eta + (F(b) - F(a))\eta < \varepsilon$.

Claim 1: There exists $N \in \mathbb{N}$ such that whenever $n, j \geq N$

$$\left| \int_I f_n dF - \int_I f_j dF \right| < \eta. \tag{11.6}$$

Proof of Claim 1: Such an N exists because, the sequence, $\left\{\int_I f_n dF\right\}_{n=1}^{\infty}$ is increasing and bounded above and is therefore a Cauchy sequence. This proves the claim.

Claim 2: There exists a sequence of disjoint sets, $\{E_n\}_{n=N}^{\infty}$ such that for $x \in E_n$, it follows that for all $j \geq n$,

$$|f_j(x) - f(x)| < \eta.$$

Proof of Claim 2: For $n \geq N$, let

$$F_n \equiv \{x : |f_j(x) - f(x)| < \eta \text{ for all } j \geq n\}.$$

Then the sets, F_n are increasing in n and their union equals I. Let $E_N = F_N$ and let $E_n = F_n \setminus F_{n-1}$. Therefore, the E_n are disjoint and their union equals I. This proves the claim.

Claim 3: There exists a gauge function δ such that if $P = \{(I_i, t_i)\}_{i=1}^{q}$ is δ fine and if

$$\mathcal{I}_j \equiv \{i : t_i \in E_j\}, \ \mathcal{I}_j \equiv \emptyset \text{ if } j < N,$$

then if $\mathcal{J} \subseteq \{n, n+1, \cdots\}$

$$\left| \sum_{j=\mathcal{J}} \sum_{i \in \mathcal{I}_j} f_n(t_i) \Delta F_i - \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}_j} \int_{I_i} f_n dF \right| \leq \eta 2^{-n}. \tag{11.7}$$

Proof of Claim 3: First note the sums are not really infinite sums because there are only finitely many tags. Thus $\mathcal{I}_j = \emptyset$ for all but finitely many j. Choose gauges, $\delta_n, n = 1, 2, \cdots$ such that if P is δ_n fine, then

$$\left| \int_I f_n dF - S(P, f_n) \right| < \eta 2^{-n} \tag{11.8}$$

and define δ on I by letting

$$\delta(x) = \min(\delta_1(x), \cdots, \delta_p(x)) \ \text{ if } x \in E_p. \tag{11.9}$$

Since the union of the E_n equals I, this defines δ on I. Note that some of the E_n may be empty. If $n \leq j$, then $\delta_n(x)$ got included in the above minimum for all $x \in E_j, j \in \mathcal{J}$. Therefore, by Corollary 11.1.14, Henstock's lemma, it follows that 11.7 holds. This proves Claim 3.

Suppose now that $P = \{(I_i, t_i)\}_{i=1}^{q}$ is δ fine. It is desired to verify 11.5. Note first

$$\int_I f_n dF = \sum_{j=\min(n,N)}^{\infty} \sum_{i \in \mathcal{I}_j} \int_{I_i} f_n dF, \ S(P, f_n) = \sum_{j=\min(n,N)}^{\infty} \sum_{i \in \mathcal{I}_j} f_n(t_i) \Delta F_i$$

where as before, the sums are really finite. Recall $\mathcal{I}_j \equiv \emptyset$ if $j < N$. If $n \leq N$, then by Claim 3,

$$\left| \int_I f_n dF - S(P, f) \right| = \left| \sum_{j=n}^{\infty} \sum_{i \in \mathcal{I}_j} \int_{I_i} f_n dF - \sum_{j=n}^{\infty} \sum_{i \in \mathcal{I}_j} f_n(t_i) \Delta F_i \right| < \eta 2^{-n} < \varepsilon$$

Therefore, 11.5 holds in case $n \leq N$.

The interesting case is when $n > N$. Then in this case,

$$\left| \int_I f_n dF - S(P, f) \right| = \left| \sum_{j=N}^{\infty} \sum_{i \in \mathcal{I}_j} \int_{I_i} f_n dF - \sum_{j=N}^{\infty} \sum_{i \in \mathcal{I}_j} f_n(t_i) \Delta F_i \right|$$

$$\leq \left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_n dF - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_n(t_i)\Delta F_i\right| + \left|\sum_{j=n}^{\infty}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_n dF - \sum_{j=n}^{\infty}\sum_{i\in\mathcal{I}_j} f_n(t_i)\Delta F_i\right| \quad (11.10)$$

By Claim 3, the last term in 11.10 is smaller than $\eta 2^{-n}$. Thus

$$\left|\int_I f_n dF - S(P,f)\right| \leq \eta 2^{-n} + \left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_n dF - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_n(t_i)\Delta F_i\right|$$

$$\leq \eta 2^{-n} + \left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_n dF - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_j dF\right| \quad (11.11)$$

$$+ \left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_j dF - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_j(t_i)\Delta F_i\right| \quad (11.12)$$

$$+ \left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_j(t_i)\Delta F_i - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_n(t_i)\Delta F_i\right| \quad (11.13)$$

Then using Claim 1 on 11.11, the messy term is dominated by

$$\left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_n dF - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_j dF\right|$$

$$\leq \left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\left(\int_{I_i} f_n dF - \int_{I_i} f_j dF\right)\right|$$

$$= \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i}(f_n - f_j)\, dF \leq \left|\int_I f_n dF - \int_I f_j dF\right| < \eta$$

Next consider the term in 11.12. By Claim 3 with $\mathcal{J} \equiv \{j\}$

$$\left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j}\int_{I_i} f_j dF - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_j(t_i)\Delta F_i\right|$$

$$\leq \sum_{j=N}^{n-1}\left|\sum_{i\in\mathcal{I}_j}\int_{I_i} f_j dF - \sum_{i\in\mathcal{I}_j} f_j(t_i)\Delta F_i\right|$$

$$\leq \sum_{j=N}^{n-1}\eta 2^{-j} \leq \eta.$$

Finally consider 11.13. By Claim 2,

$$\left|\sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_j(t_i)\,\Delta F_i - \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} f_n(t_i)\,\Delta F_i\right|$$
$$\leq \sum_{j=N}^{n-1}\sum_{i\in\mathcal{I}_j} |f_j(t_i) - f_n(t_i)|\,\Delta F_i \leq \eta\,(F(b) - F(a))$$

Thus the terms in 11.11 - 11.13 have been estimated and are no larger than

$$\eta 2^{-n} + \eta + \eta + \eta\,(F(b) - F(a)) < \varepsilon$$

by the choice of η. This proves the lemma.

Now here is the proof of the monotone convergence theorem. By the lemma there exists a gauge, δ, such that if P is δ fine then

$$\left|\int_I f_n dF - S(P, f_n)\right| < \varepsilon$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality,

$$\left|\lim_{n\to\infty}\int_I f_n dF - S(P, f)\right| \leq \varepsilon$$

and since ε is arbitrary, this shows that

$$\int_I f dF = \lim_{n\to\infty}\int_I f_n dF.$$

This proves the theorem.

Example 11.1.19 *Let*

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Then $f \in R^*[0,1]$ *and*

$$\int_0^1 f dF = 0.$$

Here assume $F(x) = x$.

This is obvious. Let $f_n(x)$ to equal 1 on the first n rational numbers in an enumeration of the rationals and zero everywhere else. Clearly $f_n(x) \uparrow f(x)$ for every x and also f_n is Riemann integrable and has integral 0. Now apply the monotone convergence theorem. Note this example is one which has no Riemann or Darboux integral.

11.2 Integrals Of Derivatives

Consider the case where $F(t) = t$. Here I will write dt for dF. The generalized Riemann integral does something very significant which is far superior to what can be achieved with other integrals. It can always integrate derivatives. Suppose f is defined on an interval, $[a,b]$ and that $f'(x)$ exists for all $x \in [a,b]$, taking the derivative from the right or left at the endpoints. What about the formula

$$\int_a^b f'(t)\,dt = f(b) - f(a)? \tag{11.14}$$

Can one take the integral of f'? If f' is continuous there is no problem of course. However, sometimes the derivative may exist and yet not be continuous. Here is a simple example.

Example 11.2.1 *Let*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}.$$

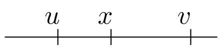
You can verify that f has a derivative on $[0,1]$ but that this derivative is not continuous.

The fact that derivatives are generalized Riemann integrable depends on the following simple lemma called the straddle lemma by McLeod [26].

Lemma 11.2.2 *Suppose $f : [a,b] \to \mathbb{R}$ is differentiable. Then there exist $\delta(x) > 0$ such that if $u \le x \le v$ and $u, v \in [x - \delta(x), x + \delta(x)]$, then*

$$|f(v) - f(u) - f'(x)(v-u)| < \varepsilon |v - u|.$$

Proof: Consider the following picture.



From the definition of the derivative, there exists $\delta(x) > 0$ such that if $|v - x|$, $|x - u| < \delta(x)$, then

$$|f(u) - f(x) - f'(x)(u - x)| < \frac{\varepsilon}{2} |u - x|$$

and

$$|f'(x)(v - x) - f(v) + f(x)| < \frac{\varepsilon}{2} |v - x|$$

Now add these and use the triangle inequality along with the above picture to write

$$|f'(x)(v - u) - (f(v) - f(u))| < \varepsilon |v - u|.$$

This proves the lemma.

The next proposition says 11.14 makes sense for the generalized Riemann integral.

Proposition 11.2.3 *Suppose $f : [a,b] \to \mathbb{R}$ is differentiable. Then $f' \in R^*[a,b]$ and*

$$f(b) - f(a) = \int_a^b f' dx$$

where the integrator function is $F(x) = x$.

Proof: Let $\varepsilon > 0$ be given and let $\delta(x)$ be such that the conclusion of the above lemma holds for ε replaced with $\varepsilon/(b-a)$. Then let $P = \{(I_i, t_i)\}_{i=1}^n$ be δ fine. Then using the triangle inequality and the result of the above lemma with $\Delta x_i = x_i - x_{i-1}$,

$$\begin{aligned} \left| f(b) - f(a) - \sum_{i=1}^n f'(t_i) \Delta x_i \right| &= \left| \sum_{i=1}^n f(x_i) - f(x_{i-1}) - f'(t_i) \Delta x_i \right| \\ &\le \sum_{i=1}^n \varepsilon/(b-a) \Delta x_i = \varepsilon. \end{aligned}$$

This proves the proposition.

With this proposition there is a very simple statement of the integration by parts formula which follows immediately.

Corollary 11.2.4 *Suppose f, g are differentiable on $[a,b]$. Then $f'g \in R^*[a,b]$ if and only if $g'f \in R^*[a,b]$ and in this case,*

$$fg|_a^b - \int_a^b fg'dx = \int_a^b f'gdx$$

The following example, is very significant. It exposes an unpleasant property of the generalized Riemann integral. You can't multiply two generalized Riemann integrable functions together and expect to get one which is generalized Riemann integrable. Also, just because f is generalized Riemann integrable, you cannot conclude $|f|$ is. This is very different than the case of the Riemann integrable. It is unpleasant from the point of view of pushing symbols. The reason for this unpleasantness is that there are so many functions which can be integrated by the generalized Riemann integral. When you say a function is generalized Riemann integrable, you do not say much about it.

Example 11.2.5 *Consider the function,*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Then $f'(x)$ exists for all $x \in \mathbb{R}$ and equals

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Then f' is generalized Riemann integrable on $[0,1]$ because it is a derivative. Now let $\psi(x)$ denote the sign of $f(x)$. Thus

$$\psi(x) \equiv \begin{cases} 1 \text{ if } f(x) > 0 \\ -1 \text{ if } f(x) < 0 \\ 0 \text{ if } f(x) = 0 \end{cases}$$

Then ψ is a bounded function and you can argue it is Riemann integrable on $[0,1]$. However, $\psi(x) f(x) = |f(x)|$ and this is not generalized Riemann integrable.

11.3 Exercises

1. Prove that if $f_n \in R^*[a,b]$ and $\{f_n\}$ converges uniformly to f, then $f \in R^*[a,b]$ and $\lim_{n\to\infty} \int_I f_n = \int_I f$.

2. In Example 11.2.5 there is the function given

$$g(x) \equiv \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

   It equals the derivative of a function as explained in this example. Thus g is generalized Riemann integrable on $[0,1]$. What about the function, $h(x) = \max(0, g(x))$?

3. Let $f \in R^*[a,b]$ and consider the function, $x \to \int_a^x f(t)\,dt$. Is this function continuous? Explain. **Hint:** Let $\varepsilon > 0$ be given and let a gauge, δ be such that if P is δ fine then

$$\left|S(P,f) - \int_a^b f dx\right| < \varepsilon/2$$

Now pick $h < \delta(x)$ for some $x \in (a,b)$ such that $x + h < b$. Then consider the single tagged interval, $([x, x+h], x)$ where x is the tag. By Corollary 11.1.14

$$\left| f(x) h - \int_x^{x+h} f(t)\, dt \right| < \varepsilon/2.$$

Now you finish the argument and show f is continuous from the right. A similar argument will work for continuity from the left.

4. Generalize Problem 3 to the case where the integrator function is more general. You need to consider two cases, one when the integrator function is continuous at x and the other when it is not continuous.

5. Suppose $f \in R^*[a,b]$ and f is continuous at $x \in [a,b]$. Show $G(y) \equiv \int_a^y f(t)\, dt$ is differentiable at x and $G'(x) = f(x)$.

6. Suppose f has $n+1$ derivatives on an open interval containing c. Show using induction and integration by parts that

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^k(c)}{k!}(x-c)^k + \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\, dt.$$

Would this technique work with the ordinary Riemann or Darboux integrals?

7. The ordinary Riemann integral is only applicable to bounded functions. However, the Generalized Riemann integral has no such restriction. Let $f(x) = x^{-1/2}$ for $x > 0$ and 0 for $x = 0$. Find $\int_0^1 x^{-1/2} dx$. **Hint:** Let $f_n(x) = 0$ for $x \in [0, 1/n]$ and $x^{-1/2}$ for $x > 1/n$. Now consider each of these functions and use the monotone convergence theorem.

8. Can you establish a version of the monotone convergence theorem which has a decreasing sequence of functions, $\{f_k\}$ rather than an increasing sequence?

9. For E a subset of $\mathbb{R}$, let $\mathcal{X}_E(x)$ be defined by

$$\mathcal{X}_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}$$

For F an integrator function, define E to be measurable if for all $n \in \mathbb{N}$, $\mathcal{X}_E \in R^*[-n,n]$ and in this case, let

$$\mu(E) \equiv \sup\left\{ \int_{-n}^{n} \mathcal{X}_E(t)\, dt : n \in \mathbb{N} \right\}$$

Show that if each E_k is measurable, then so is $\cup_{k=1}^{\infty} E_k$ and if E is measurable, then so is $\mathbb{R} \setminus E$. **Hint:** This will involve the monotone convergence theorem.

10. The gamma function is defined for $x > 0$ as

$$\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1} dt \equiv \lim_{R\to\infty} \int_0^R e^{-t} t^{x-1} dt$$

Show this limit exists. Also show that

$$\Gamma(x+1) = x\Gamma(x),\ \Gamma(1) = 1.$$

How does $\Gamma(n)$ for n an integer compare with $(n-1)!$?

11. This problem outlines a treatment of Stirling's formula which is a very useful approximation to $n!$ based on a section in [30]. It is an excellent application of the monotone convergence theorem. Follow and justify the following steps. The impoper integrals are defined as in Problem 10. Here $x > 0$.

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt$$

First change the variables letting $t = x(1+u)$ to get $\Gamma(x+1) =$

$$e^{-x} x^{x+1} \int_{-1}^\infty \left(e^{-u}(1+u)\right)^x du$$

Next make the change of variables $u = s\sqrt{\frac{2}{x}}$ to obtain $\Gamma(x+1) =$

$$\sqrt{2} e^{-x} x^{x+(1/2)} \int_{-\sqrt{\frac{x}{2}}}^\infty \left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^x ds$$

The integrand is increasing in x. This is most easily seen by taking ln of the integrand and then taking the derivative with respect to x. This derivative is positive. Next show the limit of the integrand is e^{-s^2}. This isn't too bad if you take the ln and then use L'Hospital's rule. Consider the integral. Explain why it must be increasing in x. Then use Lemma 5.4.3 on Page 82 which is about interchanging the order of sup and the monotone convergence theorem. Tell where it is used and explain why it can be used in the following computation. Remember, this theorem pertains to a sequence of functions.

$$\begin{aligned}
&\lim_{x\to\infty} \int_{-\sqrt{\frac{x}{2}}}^\infty \left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^x ds \\
=\ & \sup_{x>0} \int_{-\sqrt{\frac{x}{2}}}^\infty \left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^x ds \\
=\ & \sup_{x>0}\sup_{y>0}\sup_{z>0} \int_{-z}^y \mathcal{X}_{\left[-\sqrt{\frac{x}{2}},y\right]}(s) \left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^x ds \\
=\ & \sup_{y>0}\sup_{z>0}\sup_{x>0} \int_{-z}^y \mathcal{X}_{\left[-\sqrt{\frac{x}{2}},y\right]}(s) \left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^x ds \\
=\ & \sup_{y>0}\sup_{z>0}\lim_{x\to\infty} \int_{-z}^y \mathcal{X}_{\left[-\sqrt{\frac{x}{2}},y\right]}(s) \left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^x ds \\
=\ & \sup_{y>0}\sup_{z>0} \int_{-z}^y e^{-s^2} ds \equiv \int_{-\infty}^\infty e^{-s^2} ds
\end{aligned}$$

Now Stirling's formula is

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2} e^{-x} x^{x+(1/2)}} = \int_{-\infty}^\infty e^{-s^2} ds$$

where this last improper integral equals $\sqrt{\pi}$ thanks to Problem 29 on Page 214.

12. To show you the power of Stirling's formula, find whether the series

$$\sum_{n=1}^\infty \frac{n! e^n}{n^n}$$

converges. The ratio test falls flat but you can try it if you like. Now explain why, if n is large enough

$$n! \geq \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-s^2} ds\right) \sqrt{2} e^{-n} n^{n+(1/2)} \equiv c\sqrt{2} e^{-n} n^{n+(1/2)}$$

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Index